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Structural reliability using finite element method

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ABSTRACT

Structural Reliability Using Finite Element Method

by

Che-Chen Liou

During the last decade, structural reliability theory has been treated in a large number of research papers; therefore, from being a subject only well known by a relatively small number of researchers, it has become an important engineering discipline. From the application point of view, many practical applications have been made successfully.

In this dissertation some important fundamental concepts in statistics and in reliability theory are presented. The concept of failure mode can be defined as: A set of failed elements turn a structure into a mechanism. Usually, a structure has many possible failure modes; therefore, it will be necessary to estimate the reliability with respect to each specified failure mode, and then to estimate the overall reliability of the structure from a system point of view. In this dissertation the methodology of using ANSYS finite element software to identity the failure modes is introduced in detail.

The modelling used in this dissertation is based on the assumption that the total reliability of the structure can be sufficiently accurately estimated by considering only a finite number of significant failure modes and then combining them in a complex reliability system. Usually, the reliability of a structural system is modeled by a series of failure modes each composed of failure elements assembled in parallel.

STRUCTURAL RELIABILITY USING FINITE ELEMENT METHOD

by
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To my parents and my wife

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CHAPTER 1

INTRODUCTION

During the last decade structural reliability theory has been used in a large number of research reports and conference papers; therefore, from being a subject only well known by a relatively small number of researchers, it now becomes an important engineering discipline. From the application point of view, structural reliability is a relatively new area. However, many practical applications have been made successfully.

In this thesis, some important fundamental concepts in statistics and in reliability theory are presented. In the process of solving a real-life problem in statistics, three steps must be considered. First, a mathematical model is selected. Second, a check is made of the reasonableness of the model. Third, an appropriate conclusion is obtained from this model to solve the specified problem. The theory of statistics can be treated as a fundamental part of reliability analysis in which probability is the basic tool.

The reliability of a structure in this thesis can be defined as: the ability of a structure to perform its design purpose, under some specified conditions, for a reasonably accepted probability of failure. The reliability of a structure is denoted by R and is defined as $R = 1 - P_f$ where P_f is the probability of failure of the structure. An individual's approach to probability depends on the nature of one's interest in the subject. The applied engineers usually think of probability as the proportion of times that a certain event will occur if the experiment related to the event is repeated indefinitely. Some statisticians think of probability of a system as its ability to perform a required function, under stated conditions, for a stated period of time.The probability should be a number between zero and one. The term reliability is also used to denote the probability of success.

In evaluating structural reliability, the first step is usually to identify the variables by which the reliability of the structure can be described. Typically, these variables include material strengths, geometrical quantities, and external loads; These variables are called basic variables and are modelled as random variables or as stochastic process, but only those modelled by random variables are considered in this thesis. Usually, a structure has many possible failure modes; therefore, we will usually estimate the reliability with respect to each specified failure mode, and then estimate the overall reliability of the structure from a system point of view. A failure mode can be defined as: A set of failure members forms a mechanism which causes the structure to fail, then this set of failure members is called failure mode. A failure mode can be represented by a parallel system.

It is a common recognition that an estimate of the reliability of a structure must be based on a system approach. Sometimes it is sufficient to estimate the individual reliability of each member in a structure: for instance, for a statically determinate structure where failure in any member will result in the failure of the whole structure. However, failure of a single member will not always result in failure of the whole structure because the existing members may be able to sustain the external loads by redistribution of the internal load effects. For instance, a statically indeterminate (redundant) structure, where failure of the structure needs that more than one member fail.

In practice, a structure is usually so complex that the number of possible different failure modes is so large that they can not all be taken into account; therefore, the model must be built up carefully so that the most significant failure modes of the structure are chosen in the model. In order to assess the reliability of a structure, those failure modes and their safety margins must be given. For a simple structure the safety margins can be obtained by hand calculation. In the conventional analysis the failure modes and their safety margines are derived by using the priciple of virtual work. However, it is very difficultto derive significant failure modes for a large redundant structure. In this thesis we will describe how to use ANSYS (a finite element software) to derive the failure modes in detail and the modelling used in this thesis is based on the assumption that the total reliability of the structure can be sufficiently accurately estimated by considering only a finite number of significant failure modes and then combining them in a complex reliability system.

Usually, it is assumed that the reliability of a structure is estimated on the basis of a series system modelling, where the components are failure modes, and the failure modes are modelling by parallel systems. When the reliability of a structural system is modelled by a series system of parallel systems, the reliability of the structure can be estimated by the following steps: the first step is to calculate the probability of failure for each parallel system, the second step is to evaluate the correlationship between the parallel systems, and the final step is to calculate the probability of failure of the series system.

For some structures, the reliabilities of structures are calculated on the basis of failure of single components, where the probability of failure of any component and the correlation between failed components are taken into account. Then all the failure components are combined to make up the series system. Modelling of this type is called system reliability at level 1. The evaluation of the structural reliability can be obtained with satisfactory accuracy by only including failure components with high probabilities of failure. Such significant failure components can be selected by choosing those failure components with P_f values in an interval $[P_{fmax}, P_{fmax} - \Delta P_f]$, where ΔP_f must be chosen properly.

For some structures,the reliabilities of the structures are calculated on the basis of failure of two faillure components, where the probability of failure of any pair of failure components and the correlation between failure pairs are taken into account, and then all the failure pairs are combined to make up the series system. The modelling of this type is called system reliability at level 2. The evaluation of structural reliability can be obtained with satisfactory accuracy by only including failure pairs with high probability of failure.

To obtain the so-called significant pairs of failure components, the structure is modified by assuming failure in the significant failure components and applying artificial loads which are the strength capacities of the failure components if the components are ductile. No artificial loads are applied if the failure components are brittle. Then the modified structure is analysed elastically and new P_f values are calculated for all surving components. Surviving components with high P_f values are then combined with the significant failure components so that the significant pairs of failure components can be determined. By continuing in the same procedure, system reliability at level N, N=3,4,5...... can be defined.

The most frequently used modelling of system reliability is system reliability at the mechanism level. Usually the number of mechanisms (failure modes) is very large; therefore, only some reasonable number of significant mechanisms should be considered. The procedure described above can be continued until formation of mechanisms, but when a structure is very complex it is better to base the ANSYS reliability analysis on the fundamental mechanisms and on the linear combinations of fundamental mechanisms.

In order to assess the reliability of a structure, the failure modes and safety margins must be given. Automatic generation of failure modes was initiated by using an incremental method suggested by Moses, F., 1983. A method for generation basic mechanism was proposed by Watwood, V.B., 1979. A general procedure for expressing the safety margins in terms of the random variables was developed by Murotsu, Y., 1980.

During last decade, many papers have been published, but in most of these papers it is only shown how the reliability of single structural members can be evaluated. Some of these papers have a limited scope and some are more general. In this thesis, the joint probability distribution of relevant variables is simplified and the failure criteria are idealized in such a way that the reliability evaluations can be treated for very complex structures.

The most difficult part of evaluating structural reliability is to identify the failure modes. Several methods to identify the failure modes have been suggested. In this thesis Using ANSYS finite element software to accurately identify the failure modes is describes in detail, and this method can be used extensively.

The problem of calculating the reliability of a structure is complex. The complexity is due to the large number of failure modes that have to be considered. As the structures become larger, the number of failure mechanisms grows very rapidly. Large computational resources are needed for discovering and ranking the possible failure modes. It is therefore suggested that heuristic methods should be developped in order to direct the search towards failure modes which result in the highest probability of structure failure.

This thesis is a first step towards automating the generation of all possible failure mechanisms. It is clear from the above that future work should be directed toward efficient methods of enumerating only the most significant modes of failures so that the mothods can be scaled up to tackle practical structures.

CHAPTER 2
BASIC STATISTICS

2.1 TERMINOLOGY

SAMPLE SPACE: It is a set which represents all possible outcomes to the experiment. It can be a continuous set or can be a discrete set.

EVENT (SAMPLE POINT): An event is an outcome from experiment, and an event is a subset of a sample space. An event can also be called as a sample point.

UNION: The union of two sets, say set A and B, is a new set denoted by $A \cup B$, and this new set has in it all the elements in either A or B or both.

INTERSECTION: The intersection of two sets, say set A and B, is another set denoted by $A \cap B$, and this set has in it only those elements in both A and B.

COMPLEMENT: The complement of a set is the collection of all elements in the sample space that are not in the given set. The complement of a set A is denoted by $\bar{A}$.(Assume that A and $\bar{A}$ consist of the entire sample space).

DE MORGAN'S LAW (FOR COMPLEMENTARITY):

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B}; \overline{(A \cap B)} = \bar{A} \cup \bar{B}.$$

DISJOINT (INDEPENDENT): If two sets, A and B, have no outcomes in common they are called as disjoint sets, or disjoint events.

2.2 PROBABILITY

In the applied engineers point of view, the probability can be described as the proportion of times that a certain event will occur if the experiment related to the event is repeated indefinitely. The probability should be a number between 0 and 1, and the probability of the whole sample space should be 1. Finally, if two sets A and B are disjont, the probability of the union of these two sets should be equal to the sum of the probabilities of these two sets. Therefore, the probability P should satisty:

(1) $0 \leq P(A) \leq 1$, for every set A.

(2) $P(S) = 1$, for whole sample space S.

(3) $P(A \cup B \cup \ldots) = P(A) + P(B) + \ldots$, for disjoint sets A,B,...

2.3 SOME FUNDAMENTAL PROBABILITY RULES

ADDITION RULE (OR):

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When A and $\bar{A}$ consist of the entire sample space, $P(A) = 1 - P(\bar{A})$, where $\bar{A}$ is the set that A will not occur. This formula is useful for calculating the probability that an event will occur when it is easier to calculate the probability that the event will not occur.

Addition Rule can be written in a general form:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$$

$$+ \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

CONDITIONAL RULE:

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

The probability that B will occur given that A has already occured.

MULTIPLICATION RULE (AND):

$$P(A \cap B) = P(A) P(B/A) = P(B) P(A/B)$$

If A and B are independent to each other, $P(B/A) = P(B)$, and $P(A/B) = P(A)$ .This means that the probability of B (or A) occuring is not affected by that A (or B).

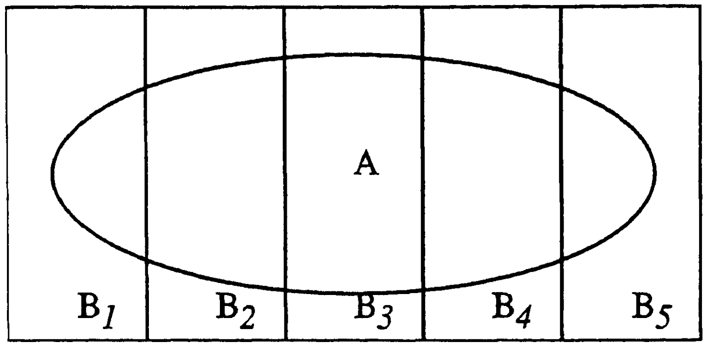
When A and B are independent, $P(A \cap B) = P(A) P(B)$

Multiplication Rule can be written in a general form

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) P(A_2/A_1) P\left(\frac{A_3}{A_1 \cap A_2}\right) \ldots$$

$$\ldots P(A_n/A_1 \cap A_2 \cap \ldots A_{n-1})$$

BAYES' FORMULA:



From the multiplication rule, it gives:

$$P(B \cap A) = P(A/B)P(B) = P(B/A)P(A)$$

From the figure shown above, the probability of A can be derived as following:

$$P(A) = P(B_1 \cap A) + P(B_2 \cap A) + \ldots + P(B_5 \cap A)$$

$$= P(A/B_1)P(B_1) + \ldots + P(A/B_5)P(B_5) = \sum_{k=1}^{5} P(A/B_k)P(B_k)$$

Therefore, $P(B_1/A) = \dfrac{P(B_1 \cap A)}{P(A)} = \dfrac{P(A/B_1)P(B_1)}{\sum_{k=1}^{5} P(A/B_k)P(B_k)}$

The result may be summarized as follows:

$$P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\sum_{j=1}^{k} P(A/B_j)P(B_j)}, i = 1, 2, \ldots, k$$

Therefore,the total probability is defined as follows:

$$P(A) = \sum_{j=1}^{\infty} P(A \cap B_j) = \sum_{j=1}^{\infty} P(A/B_j) P(B_j)$$

Where, A is an event that occured when the experiment was performed, and calculate the probability B_i that was the cause of the occurrence of A.Where, all B's are disjoint.

Example 1. There are two identical boxes; One box contains two white balls,and the second box contains one white ball and one black ball,if a box is selected randomly and one ball is drawn from it. What is the probability that the second box was selected,if the drawn ball turns out to be white ?

Let X_1 and X_2 represent box 1 and box 2 respectively, and let W represent the event of getting a white ball. Therefore, $P(X_1) = P(X_2) = \frac{1}{2}$, $P(W/X_1) = 1$, $P(W/X_2) = \frac{1}{2}$.By Bayes' formula:

$$P(X_2/W) = \frac{P(X_2) P(W/X_2)}{\sum_{i=1}^{2} P(X_i) P(W/X_i)}$$

$$= \frac{P(X_2) P(W/X_2)}{P(X_1) P(W/X_1) + P(X_2) P(W/X_2)}$$

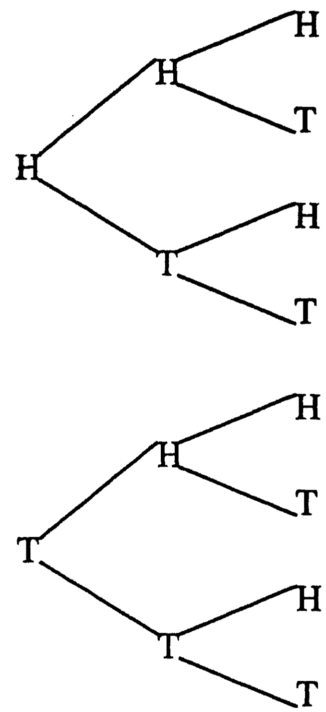
$$= \frac{\frac{1}{2}\frac{1}{2}}{\frac{1}{2}1 + \frac{1}{2}\frac{1}{2}} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

2.4 COUNTING TECHNIQUES

Sometimes, the counting of various events is very tedious unless compact counting methods are used. Some of the formulas that provide such methods are described in this section.

2.4.1 TREE

If an experiment can be treated as a multiple-stage experiment, the problem of counting sample outcomes can be considerably simplified by using the tree diagram. For example, toss a fair coin three times. This is a three-stage experiment where the various possibilities that can occur may be represented by a tree diagram as shown below: (H stands for head; T stands for tail).



Each stage of a multiple-stage experiment has as many branches as there are possibilities at that stage. Here there are two main branches for each stage. The total number of terminating points in the tree gives the all possible outcomes and therefore, the end points of a tree may be treated as the sample outcomes of a sample space corresponding to the experiment.

If there are many stages in the experiment and many possibilities at each stage, the tree will become too large to be manageable. For such problems the counting of sample outcomes can be simplified by means of other algebraic formulas.

2.4.2 PERMUTATIONS

An ordered arrangement of n different objects taken k at a time is called a *permutation* of the k objects. If two of the k objects are interchanged in their respective positions, a different permutation results.

$_nP_k = n\,(n-1)\,\ldots\,(n-k+1) = \dfrac{n!}{(n-k)\,!}$ where, $_nP_k$ is called the number of permutations of n objects taken k at a time, It is ususlly read " given n objects select k ".

2.4.3 COMBINATIONS

An unordered arrangement of n different objects taken k at a time is called a *combination* of n objects taken k at a time. Thus, if two of the k objects are interchanged in their respective positions, the same combination results.

$\binom{n}{k} = {}_nC_k = \dfrac{n!}{k!\,(n-k)\,!}$ where, $_nC_k$ is called the number of combinations of n objects taken k at a time.

2.4.4 PERMUTATIONS WHEN SOME OBJECTS ARE REPEATED

In the preceding sections, it is assumed that all the n objects are different. Sometimes, the n objects contain some similar objects. Now suppose that there are only P different kinds of objects and that there are k_1 of the first kind, k_2 of the second kind, and k_p of the p^{th} kind, where $k_1 + k_2 + \ldots + k_p = n$.

The total number of different permutations of n objects is as follows:

$$\frac{n!}{k_1!k_2!\ldots k_p!}$$

Example 2. There are four balls, two balls are white and two balls are black. What are the permutations for these four balls ? (W stands for white ball, and B stands for black ball).

$$\frac{4!}{2!\times 2!} = \frac{4\times 3\times 2\times 1}{2\times 1\times 2\times 1} = 6$$

BBWW, BWBW, WBBW, BWWB, WBWB, WWBB

2.5 RANDOM VARIABLES AND DENSITY FUNCTION

RANDOM VARIABLE: A random variable is a numerical value determined by the outcome of an experiment. Therefore, a random variable is defined on a sample space.

A sample space that contains a finite number, or an infinite sequence, of outcomes is called *discrete sample space*, while one that contains one or many intervals of outcomes is called a *continuous sample space*.

Example 3. Toss a fair coin twice, Let X = number of heads obtained, where X is a discrete random variable. S = { X | x=(0, 1, 2) }, where S is a discrete sample space.

Example 4. To measure how long a lightbulb lasts, Let X = time elapsed before burn off, Where X is a continuous random variable. S = { $0 \leq x \leq \infty$ },where S is a continuous sample space.

2.5.1 DISCRETE RANDOM VARIABLES

Let X be a discrete random variable. Then the function f(x) = P (X = x) = The probability that the random variable X assumes the value x, and f(x) is called the *discrete density function* of X.

$$P(X \in R) = \sum_{x \in R} f(x) ; P(S) = \sum_{\text{all x}} f(x) = 1$$

Where R is some set of outcomes in the sample space S, and $X \in R$ represents the event that X will assume some values in the set of R values.

The *discrete distribution function* F (x): It is a function closely related to the discrete density function f(x), and is defined as follows:

$F(x) = P(X \leq x) = \sum_{t \leq x} f(t)$,where the summation is over all those values of the random variable that are less than or equal to the specified value x.

The *discrete joint density functions*: Many experiments involve some random variables rather than one. For simplicity, consider only two discrete random variables X and Y here. A function f(x,y) gives the probability that X will be a specified value x at the same time Y will be a specified value y. This function f(x,y) that gives such probabilities is called a *discrete joint density function* of the two random variables X and Y. It can be written as:

$$f(x, y) = P(X = x, Y = y)$$

INDEPENDENT RANDOM FUNCTIONS: The random variables $X_1, X_2, \ldots, X_n$ whose joint density function is $f(x_1, x_2, \ldots, x_n)$ and whose individual density functions are $f_1(x_1), f_2(x_2), \ldots, f_n(x_n)$ are said to be independent if and only if

$f(x_1, x_2, \ldots, x_n) = f_1(x_1) f_2(x_2) \ldots f_n(x_n)$. It can be shown that functions of independent random variables are also independent random variables.

2.5.2 MARGINAL AND CONDITIONAL DISTRIBUTIONS

Consider a two variable experiment for which a random variable X will assume the value x and the second variable Y will assume the value y.

From the multiplication formula:

$$P(X \cap Y) = P(X) P(Y/X) \text{ ------ (a)}$$

Equation (a) can be expressed in terms of density function. Since $P(X \cap Y)$ gives the probability that the two random variables will assume the valures x and y, respectively,it is equivalent to f(x,y). Similarly, P(Y/X) is equivalent to f(y/x); P(Y) is equivalent to f(y), and P(Y/X) is equivalent to f(y/x).

Therefore, equation (a) can be written as:

$$f(x, y) = f(x) f(y/x) \text{ ------ (b)}$$

Since f(y/x) is the conditional probability that Y will assume the value y when X is a fixed value x, the sum of f(y/x) over all possible values of y for this fixed value of x must be equal to 1. Therefore, if both sides of (b) are summed over all possible values of y, one can get:

X MARGINAL DENSITY FUNCTION: $f(x) = \sum_{\text{all y}} f(x, y)$

In connection with the joint density function f(x,y), the function f(x) is only the density function of X.

Similarly, one can get:

Y MARGINAL DENSITY FUNCTION: $h(y) = \sum_{\text{all x}} f(x, y)$

These results show that one can get any density function from the joint density function of two random variables and it only sums the joint density function over all values of the other variable.

From equation (b),one can obtain:

CONDITIONAL DENSITY FUNCTION of Y for X given:

$$f(y/x) = \frac{f(x, y)}{f(x)}$$

In a similar manner, one can obtain:

CONDITIONAL DENSITY FUNCTION of X for Y given:

$$f(x/y) = \frac{f(x, y)}{f(y)}$$

These results show that, if the joint density function of two random variables is given, the conditional density function can be obtained by dividing the joint density function by the density function of the given variable.

2.5.3 CONTINUOUS RANDOM VARIABLES

For a continuous random variable X, its corresponding density function possesses the properties as follows:

(1) $f(x) \geq 0$

(2) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(3) $\int_{a}^{b} f(x)\, dx = P(a \leq x \leq b)$

(4) $f(x, y) = \frac{\partial^2}{\partial x \partial y} F(x, y)$

(5) $F(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s, t)\, ds dt$

Therefore, probabilities for continuous variables are always calculated by integrals, and those for discrete variables are given by sums.

THE DISTRIBUTION FUNCTION F(x) for the continuous variable X is defined by:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\, dt$$

Sometimes, it is easier to find the distribution function of a random variable. After the distribution function has been found, the density function can be obtained by differentiating the distribution function. $\frac{d}{dx} F(x) = f(x)$

A density function f(x,y) of two continuous random variables X and Y represents geometrically a surface in three dimensions, just as a density function f(x) of one random variable represents a curve in two dimensions. The integrals of f(x,y) can produce probabilities, and the total volume under this surface should be equal to 1. Therefore, a joint density function of two continuous random variables X and Y will possess the following properties:

(1) $f(x, y) \geq 0$

(2) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dxdy = 1$

(3) $\int_{a}^{b} \int_{c}^{d} f(x, y)\, dxdy = P\ (c < X < d, a < Y < b)$

If two continuous random variables are unrelated they are said to be independent of each other. The definition of independence is defined by: $f(x, y) = f(x) f(y)$

By using integrals in place of sums, formulas can be derived for marginal and conditional density functions just as in the cases of discrete variables.

X MARGINAL DENSITY FUNCTION: $f(x) = \int_{\text{all y}} f(x, y)\, dy$

Y MARGINAL DENSITY FUNCTION: $h(y) = \int_{\text{all x}} f(x, y)\, dx$

CONDITIONAL DENSITY FUNCTION: $f(y/x) = \dfrac{f(x, y)}{f(x)}$

CONDITIONAL DENSITY FUNCTION: $f(x/y) = \dfrac{f(x, y)}{f(y)}$

CHAPTER 3
SOME PARTICULAR PROBABILITY DISTRIBUTIONS USED IN RELIABILITY

3.1 INTRODUCTION

This chapter will present some properties of the main probability distributions used in reliability. Some distributions are for discrete variables, some are for continuous variables. In each case the probability distribution will be defined by its probability density function. In many problems, it suffices to consider certain low order moments of a distribution rather than to study the entire distribution. Therefore, this chapter will introduce the moments of particular distributions and density functions.

3.2 DISCRETE DISTRIBUTIONS

3.2.1 EXPECTATION, MOMENT, AND MOMENT GENERATING FUNCTIONS

EXPECTATION: The expected value of the function h(X) of the discrete random variable X, whose density is f(x), is defined by:

$$E[h(X)] = \mu = \sum_{i=1}^{\infty} h(x_i) f(x_i)$$

The expected value of the random variable is usually called the *mean or mean value* of the random variable.

MOMENTS:

The kth moment of the discrete random variable X, whose density is f(x) is defined by:

$$E[X^k] = \mu'_k = \sum_{i=1}^{\infty} x_i^k f(x_i)$$

The first moment μ'_1 will be used so often that it is given a special symbol μ

The kth moment about the mean of the distribution of the discrete random variable X whose density function is f(x) is defined by:

$$\mu_k = E[(X-\mu)^k] = \sum_{i=1}^{\infty} (x_i - \mu)^k f(x_i)$$

The moments of a distribution are very useful for describing a distribution when the density function is not available.

Usually, only the first two moments are used often to describe two important properties of the distribution. The first moment, μ, is used to determine where is the center of the distribution, and the second moment about mean, μ_2, is used to determine the degree of concentration of the distribution about mean. Since the second moment about mean is used so often it is denoted by a special symbol σ^2 and is called the *variance* of the distribution.

The square root of the variance is called the *standard deviation* of the distribution and is denoted by symbol σ.

Sometimes, it is convenient to evaluate variance, σ^2, by evaluating the first two moments about the origin and then calculate σ^2 from them rather then calculate it directly.

$$\mu_2 = \sigma^2 = E[(X-\mu)^2] = \sum_{i=1}^{\infty}(x_i-\mu)^2 f(x_i) = \mu'_2 - \mu^2$$

MOMENT GENERATING FUNCTION:

The moment generating function is a function that can generate moments. The moment generating function of the discrete random variable X whose density function is f(x) is defined by:

$$M_X(\theta) = E[e^{\theta X}] = \sum_{i=1}^{\infty} e^{\theta x_i} f(x_i) = \sum_{i=1}^{\infty}\left[1+\theta x_i+\frac{\theta^2 x_i^2}{2!}+\ldots\right] f(x_i)$$

$$= \sum_{i=1}^{\infty} f(x_i) + \theta\sum_{i=1}^{\infty} x_i f(x_i) + \frac{\theta^2}{2!}\sum_{i=1}^{\infty} x_i^2 f(x_i) + \ldots$$

$= 1+\theta\mu'_1+\frac{\theta^2}{2!}\mu'_2+\ldots$ This series is a function of parameter θ only.

If a particular moment is desired, it may be obtained by evaluating the proper derivative of $M_X(\theta)$ at $\theta = 0$.

$$\mu'_k = \left.\frac{d^k M}{d\theta^k}\right|_{\theta = 0}$$

3.2.2SOME ALGEBRA FOR RANDOM VARIABLES

From the basic definitions, it is easy to derive the following relations which are true for a constant c:

(1) $E[cX] = cE[X]$

(2) $E[c+X] = c+E[X]$

(3) $VAR[cX] = c^2 VAR[X]$

(4) $VAR[c+X] = VAR[X]$

(5) $VAR[X] = E[X^2] - (E[X])^2$

(6) $E[X+Y] = E[X] + E[Y]$

(7) $COV[X,Y] = E[(X-\mu_X)(Y-\mu_Y)] = E[XY] - E[X]E[Y]$

$$= \sum_{i=1}^{\infty} (x_i - \mu_x)(y_i - \mu_y) f(x_i, y_i)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f(x,y)\, dxdy$$

(8) $VAR[X+Y] = VAR[X] + VAR[Y] + 2[(X-\mu_X)(Y-\mu_Y)]$

$$= VAR[X] + VAR[Y] + 2COV[X,Y]$$

(9) $\rho_{XY} = \dfrac{COV[X,Y]}{\sigma_X \sigma_Y}, -1 \le \rho_{XY} \le 1$

(10) $$VAR[X-Y] = VAR[X] + VAR[Y] - 2COV[X,Y]$$

(11) $$E[XY] = E[X]E[Y] + COV[X,Y]$$

When X and Y are non-correlated :

(12) $$VAR[X+Y] = VAR[X] + VAR[Y]$$

(13) $$E[X-Y] = E[X] - E[Y]$$

Where COV[.] stands for the covariance of X and Y, ρ_{XY} stands for the correlation coefficient, VAR[.] stands for variance, and E[.] stands for mean. If Cov[X,Y]=0,then X and Y are uncorrelated.Independent must be uncorrelated(inverse not applied).

3.2.3BINOMIAL DISTRIBUTION

Consider an experiment which consists of n trials; each trial has only two possible outcomes $(A, \bar{A})$. Let p be the probability of A, and q = 1 - p be the probability of $\bar{A}$.The trials are assumed to be independent so p doesn't change. The discrete random variable X representing the number of occurrences of event A in the n trials is binomially distributed with parameters (p,n):

$$f(x) = P(X=x) = \frac{nPn}{x!(n-x)!}p^x q^{n-x} = \binom{n}{x} p^x q^{n-x}, x=0,1,2,...,n.$$

The corresponding distribution function is:

$$F(X) = P(X \le x) = \sum_{i=0}^{x} \binom{n}{x} p^x q^{n-x}, \text{and}$$

$$\mu = E[X] = np; VAR[X] = npq$$

3.2.4POISSON DISTRIBUTION:

Poisson density function is an approximation to the binomal density function. This approxition may be applied when the number of trials is large or the probability p is very small. The probability density function of Poisson distribution is defined as follows:

$$f(x) = \frac{e^{-\mu}\mu^{x}}{x!}$$

The corresponding distribution function is:

$$F(x) = \sum_{i=0}^{x} \frac{e^{-\mu}\mu^{x}}{x!}, \text{and}$$

$$E[X] = \mu; VAR[X] = \mu$$

3.3COUNTINUOUS DISTRIBUTIONS

In this section we will briefly consider some main distributions used in reliability. Since it is necessary to calculate the moments of these distributions, the definition of the k^{th} moment of a continuous random variable is discussed first.

3.3.1EXPECTION, MOMENT, AND MOMENT GENERATING FUNCTION

EXPECTATION: The expected value of the function h(X) of the continuous random variable X, whose density is f(x) is defined by:

$$E[h(X)] = \mu = \int_{-\infty}^{\infty} h(x) f(x)\, dx$$

MOMENTS: The k^{th} moment of the continuous random variable X, whose density is f(x) is defined by:

$$E[X^k] = \mu'_k = \int_{-\infty}^{\infty} x^k f(x)\, dx$$

The k^{th} moment about the mean is defined by:

$$E[(X-\mu)^k] = \mu_k = \int_{-\infty}^{\infty} (x-\mu)^k f(x)\, dx$$

Usually, only the first two moments are used:

(1) $$\mu'_1 = \mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

(2) $$\mu_2 = \sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx = \mu'_2 - \mu^2$$

MOMENT GENERATING FUNCTION:

The moment generating function of the continuous random variable X whose density function is f(x) is defined by:

$$M_X(\theta) = E[e^{\theta X}] = \int_{-\infty}^{\infty} e^{\theta x} f(x)\, dx$$

SOME MAIN DISTRIBUTIONS USED IN RELIABILITY AND THEIR CORRESPONDING MOMENTS:

DISTRIBUTIONS:	DENSITY FUNCTION:	MEAN	VARIANCE:
UNIFORM	$f(x) = \begin{cases} \frac{1}{(b-a)}, a \le x \le b \\ 0, elsewhere \end{cases}$	$\frac{a+b}{2}$	$\frac{1}{3}(b^2 + ab + a^2)$
NORMAL	$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$	μ	σ^2
GAMMA	$f(x) = \frac{x^{\alpha-1} e^{-(\frac{x}{\beta})}}{\beta^{\alpha}\Gamma(\alpha)}$	$\beta\alpha$	$\beta^2\alpha$
EXPONENTIAL	$f(x) = \frac{e^{-x/\beta}}{\beta}$	β	β^2
CHI-SQUARE	$f(x) = \frac{x^{(\frac{\nu}{2}-1)} e^{-(\frac{x}{2})}}{2^{\nu/2}\Gamma(\frac{\nu}{2})}$	ν	2ν
LOGNORMAL	$f(x) = \frac{1}{x\beta\sqrt{2\pi}} e^{\left[-\frac{1}{2}(\frac{lnx-\lambda}{\beta})^2\right]}$	(see section 3.3.7)	

3.3.2 UNIFORM DISTRIBUTION

The uniform distribution is the simplest continuous distribution whose density is a constant over an interval (a, b) and is zero elsewhere; therefore, its density function can be defined as follows:

$$f(x) = \begin{cases} \dfrac{1}{(b-a)}, a \leq x \leq b \\ 0, elsewhere \end{cases}$$

From the moment generating function or direct calculation, its mean value and variance can be obtained:

$$\mu = \frac{a+b}{2}$$

$$\sigma^2 = \frac{1}{3}(b^2 + ab + a^2)$$

3.3.3 NORMAL (OR GAUSSIAN) DISTRIBUTION

The normal distribution is symmetrical about the mean μ, and it is usually denoted by $N(\mu, \sigma)$. When its mean value is zero and variance is 1, the distribution $N(0, 1)$ is called a standard normal distribution.

The density function of normal distribution is defined as follows:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Therefore, the normal distribution is given by:

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} dt$$

Using the moment generating function, the mean value and the variance of the normal distribution can be obtained: $\text{mean} = \mu$, $variance = \sigma^2$.

Usually, it is convenient to use standard normal distribution instead of normal distribution, because this allows use of standard normal tables for evaluation of probability.

The standard normal density function is defined as follows:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Therefore, the standard normal distribution is given by:

$$\Phi(\beta) = \int_{-\infty}^{\beta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx$$

The relationship between normal and standard normal distributions is given by:

$$F(x) = \Phi(\beta) \text{ ,where } \beta = \frac{x-\mu}{\sigma}$$

For standard normal distribution, its mean value is equal to zero and standard distribution is equal to 1.

From the above definitions, it can be shown that a normal (or standard normal) distribution can be completely determined by specifying its mean value and standard deviation.

3.3.4GAMMA DISTRIBUTION

The gamma density function is defined by:

$$f(x) = \frac{x^{\alpha - 1} e^{-(\frac{x}{\beta})}}{\beta^{\alpha}\Gamma(\alpha)} \text{,where } \Gamma(\alpha) = \int_0^{\infty} x^{\alpha - 1} e^{-x} dx \text{ is the gamma function.}$$

Therefore, the gamma distribution is given by:

$$F(x) = \int_0^x \frac{t^{\alpha - 1} e^{-(\frac{t}{\beta})}}{\beta^{\alpha}\Gamma(\alpha)} dt = \frac{1}{\Gamma(\alpha)} \Gamma(\alpha, t/\beta) \text{ ,nd}$$

the mean value and variance of gamma distribution are given by:

$$\mu = \beta\alpha;\sigma^2 = \beta^2\alpha$$

When parameter α is an integer, it is known as the Erlangian distribution:

$$F(t) = 1 - e^{-t/\beta} \sum_{k=1}^{\alpha} \frac{(t/\beta)^{k-1}}{(k-1)!}$$

when $\alpha = 1$, the distribution becomes exponential distribution.

3.3.5EXPONENTIAL DISTRIBUTION

Exponatial distribution is a special case of gamma distribution, when $\alpha = 1$; Its density function is defined by:

$$f(x) = \frac{e^{-x/\beta}}{\beta}$$

Therefore, the exponential distribution is given by:

$$F(x) = \int_0^x \frac{e^{-t/\beta}}{\beta} dt = 1 - e^{x/\beta} \text{,and}$$

$$mean = \beta; variance = \beta^2$$

3.3.6CHI-SQUARE DISTRIBUTION

Chi-square distribution is another special case of gamma distribution, when $\beta = 2$ and $\alpha = \frac{\nu}{2}$, and its density function is defined by:

$$f(x) = \frac{x^{(\frac{\nu}{2} - 1)} e^{-(\frac{x}{2})}}{2^{\nu/2} \Gamma(\frac{\nu}{2})}$$

Therefore, the chi-square distribution is given by:

$$F(x) = \int_0^x \frac{t^{\frac{\nu}{2}-1} e^{-t/2}}{2^{\nu/2}\Gamma(\frac{\nu}{2})} dt, \text{ and}$$

$$mean = \nu, \ variance = 2\nu$$

3.3.7 LOGNORMAL DISTRIBUTION

In this distribution, the natural logrithm of the random variable X has a normal distribution, and the lognormal dendity function is defined by:

$$f(x) = \frac{1}{x\beta\sqrt{2\pi}} e^{\left[-\frac{1}{2}\left(\frac{lnx-\lambda}{\beta}\right)^2\right]}$$

Therefore, the lognormal distribution is given by:

$$F(x) = \int_{-\infty}^{x} f(t)\,dt = \Phi\left(\frac{lnx-\lambda}{\beta}\right) \text{,where} \lambda = E[lnX], \beta^2 = VAR[lnX]$$

The mean and variance are given by:

$$mean = e^{(\lambda+\frac{1}{2}\beta^2)} \text{,} variance = e^{2(\lambda+\frac{1}{2}\beta^2)}(e^{\beta^2}-1)$$

The lognormal distribution has this useful property:

$P(a < x \le b) = \Phi\left(\frac{lnb-\lambda}{\beta}\right) - \Phi\left(\frac{lna-\lambda}{\beta}\right)$; This allows use of standard normal tables for evaluation of probabilities with X lognormal.

3.3.8WEIBULL DISTRIBUTION

Its density function is defined by:

$$f(x) = \frac{\lambda}{\beta}\left(\frac{x-\alpha}{\beta}\right)^{\lambda-1} e^{\left(-\left(\frac{x-\alpha}{\beta}\right)^{\lambda}\right)};x \geq \alpha$$

When $\lambda = 1, \alpha = 0, f(x) = \frac{1}{\beta} e^{\left(-\frac{x}{\beta}\right)}$. So, exponential distribution is a special case of Weibull distribution.

$$mean = \beta\Gamma(1+\lambda) + \alpha; variance = \beta^2 (\Gamma(\frac{2}{\lambda}+1) - \Gamma^2 (1+\frac{1}{\lambda}))$$

CHAPTER 4
INTRODUCTION TO SYSTEM RELIABILITY

4.1DEFINITIONS

The reliability of a system is its ability to perform a required function,under stated conditions, for a stated period of time. The term reliability is also used to denote a probability of success or a success ratio.

RELIABILITY FUNCTION R(t): P(S will be operable during the interval [0,t]), where P(.) stands for probability measure, and S stands for system.

AVAILABILITY FUNCTION A(t): P(S will be operable at time t).For non-repairable systems A(t)=R(t).

MAINTAINABILITY FUNCTION M(t): 1-P (S will not be repaired during the interval [0, t]).

MEAN TIME TO FAILURE (MTTF): MTTF= $\int_0^\infty tf(t)\,dt = \int_0^\infty R(t)\,dt$.

Where *f(t)* is failure density function;

$$f(t) = \frac{d}{dt}F(t) = \frac{d}{dt}(1-R(t)) = -\frac{d}{dt}R(t)$$

MEAN TIME TO REPAIR (MTTR): MTTR = $\int_0^\infty (1-M(t))\,dt$

4.2REPRESENTATION OF SYSTEM LOGIC

Assume that each component has only a finite number of states; Now consider two cases:

(a) The component is normally operating (active) in a system and usually has two states: operating or failure.

(b) The component is normally non-operating (passive) in a system and only begins to operate if the main component fails. (stand-by redundency or auxiliary component).

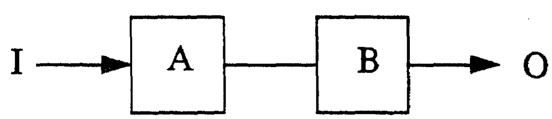
Representing the logic of a system means representing all the operating and non-operating states of the system and the connections between these various states.

There are three common methods to describe a system:

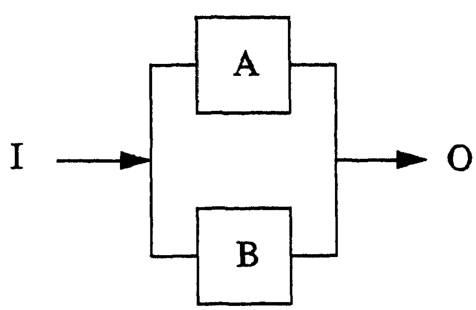
(1) RELIABILITY BLOCK DIAGRAM: (operating state)

The blocks represent the components,equipments,events,...etc.

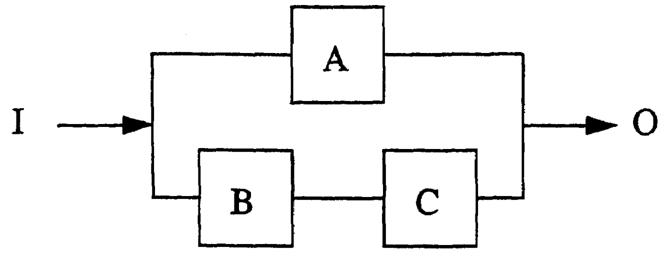
series system:



parallel system:



complex system:

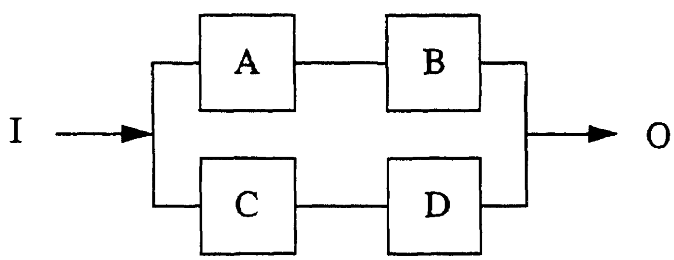


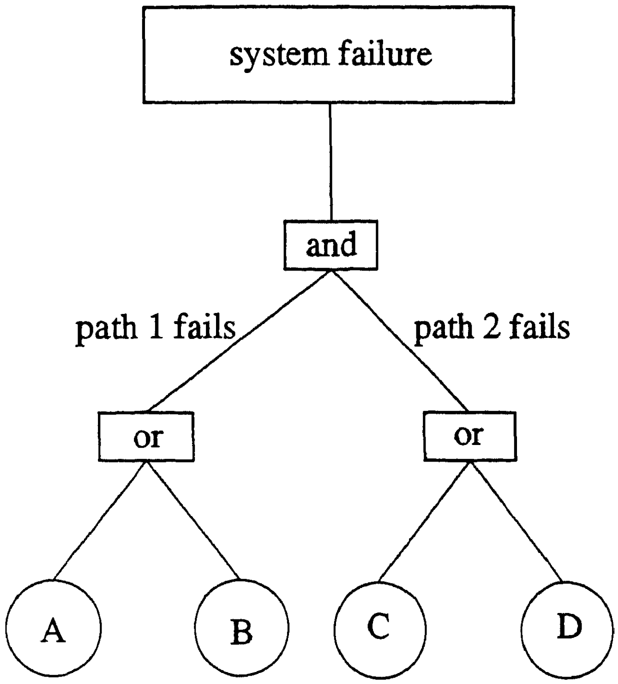
Reliability block diagram is a circuitless diagram with input and output points, and the system operates if there exists a path from input point to output point. The list of all successful paths represent all the operating states of the system.

(2) FAULT TREE: (failure state)

The starting point is a single failure event, and the failure tree provides a diagrammatic representation of the event combination resulting in the occurrence of the system failure.

Example 1. There is a system as shown below.Use the fault tree method to represent the logic of the system.





If A or B fails, path 1 fails.

If C or D fails, path 2 fails.

If path 1 and path 2 fail, the sytstem fails.

(3) MINIMAL CUT SETS:

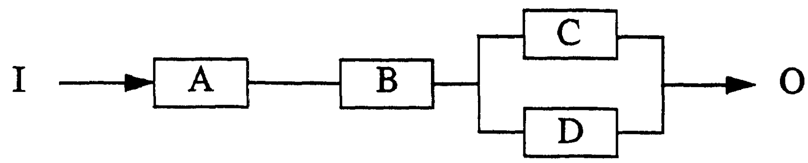
A *cut set* is a set of failure components, and this set causes the system to be in a failure state. A *minimal cut set* has the propertiy that a subset of the minimal cut set, which is also a cut set, does not exist.

HOW TO FIND THE MINIMAL CUT SETS:

(a) Fault tree approach (by Boolean function).

The minimal cut sets (or system failure functions) can be expressed by converting the fault tree to a Boolean expression, associated with each basic event.

Example 2. There is a system as shown below.Now use a Boolean function to express the minimal cut sets.



F = A + B + C D. There are three minimal cut sets: A , B, and CD.

In general, the boolean expression for a failure system can be written as:

$F = A_1 + A_2 + \ldots + A_i + \ldots + A_n$, where A_i can be the product of some basic events.

(b) Structural function (or operating function).

$$x_i = \begin{cases} 1 \text{ if } i^{th} \text{ component works} \\ 0 \text{ if } i^{th} \text{ component fails} \end{cases}$$

$$\phi(X) = \begin{cases} 1 \text{ if system works} \\ 0 \text{ if system fails} \end{cases}$$

$\phi(X) = \phi(x_1, x_2, \ldots, x_i, \ldots x_n)$, where vector $X = (x_1, x_2, \ldots, x_n)$ is the state of the system.

The structural function has the following properties:

(1) ϕ is a nondecreasing function for each variable.

i.e if $x_1 > x_2$ then $\phi(x_1) > \phi(x_2)$ or $\phi(x_1, y, z) > \phi(x_2, y, z)$

(2) $\phi(0, 0, \ldots, 0) = 0$ If all components fail then the system fails.

(3) $\phi(1, 1, \ldots, 1) = 1$ If all components work then the system works.

4.3ANALYSIS OF SIMPLE SYSTEMS

1. SERIES SYSTEM: $\phi(X) = \prod_{i=1}^{n} x_i = min(x_1, x_2, \ldots x_n)$. If any component fails then the system fails.

2. PARALLEL SYSTEM: $\phi(X) = 1 - \prod_{i=1}^{n} (1 - x_i) = max(x_1, x_2, \ldots, x_3)$.

Any component works then system works.

COMPUTE STRUCTURE FUNCTION USING MINIMAL PATH:

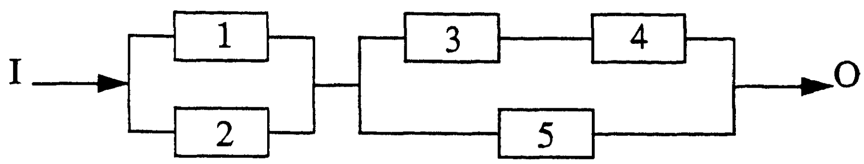
A *path set* is a set of non-failure components and this set causes the system to be in a non-failure state. A *minimal path set* has the property that a subset of the minimal path set, which is also a path set, doesn't exist.

Suppose $A_1, A_2, \ldots, A_s$ are all minimal path sets. The structure function of the j^{th} path is: $\alpha_j(X) = \min_{i \in A_j} x_i = \prod_{i \in A_j} x_i = \begin{cases} 1, & \text{all component works} \\ 0, & \text{otherwise} \end{cases}$

The system will function if at least one minimal path works (similar to a parallel system):

$$\phi(X) = \max_{1 \le j \le s} \alpha_j(X) = \max_{1 \le j \le s} \prod_{i \in A_j} x_i$$

Example 3. Compute the structure function using minimal path method, and the system is shown below:



The minimal path sets are : $\{ (1, 5), (2, 5), (1, 3, 4), (2, 3, 4) \}$

Therefore, $\phi(X) = max \; (x_1 x_5, x_2 x_5, x_1 x_3 x_4, x_2 x_3 x_4)$

$$= 1 - (1 - x_1 x_5)\,(1 - x_2 x_5)\,(1 - x_1 x_3 x_4)\,(1 - x_2 x_3 x_4)$$

COMPUTE STRUCTURE FUNCTION USING MINIMAL CUT SETS:

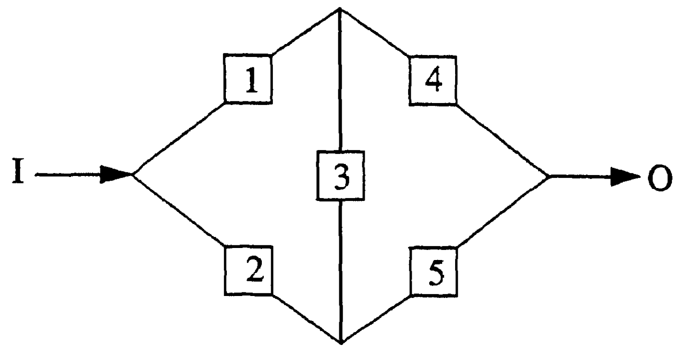
Suppose $B_1, B_2, \ldots, B_n$ are all minimal cut sets. The structure function of th j^{th}cut set is:

$$\beta_j(X) = \max_{i \in \beta_j} x_i = \begin{cases} 1, & \text{if at least one component in } j^{th} \text{ cut is functioning} \\ 0, & \text{otherwise (similar to a parallel system)} \end{cases}$$

The system will be functioning if each cut $\beta_j(X) = 1$ is functioning, (similar to a series system).

$$\phi(X) = \min_{1 < j < k} \beta_j(X) = \prod_{j=1}^{k} \beta_j(X) = \prod_{j=1}^{k} \max_{i \in \beta_j} x_i \; .$$

Example 4. Compute the structure function using the minimal cut set method, and the system is shown below:



The minimal cut sets are: { (1,2),(4,5),(1,3,5),(2,3,4) }

$$\phi(X) = max\ (x_1, x_2)\, max\ (x_4, x_5)\, max\ (x_1, x_3, x_5)\, max\ (x_2, x_3, x_4)$$

$$= [1-(1-x_1)(1-x_2)]\,[1-(1-x_4)(1-x_5)]$$

$$\times [1-(1-x_1)(1-x_3)(1-x_5)]\,[1-(1-x_2)(1-x_3)(1-x_5)]$$

4.4RELIABILITY OF SIMPLE SYSTEMS

Reliabilityfunction

$$R(p) = P\{\phi(X=1)\} = E[\phi(X)] = R(p_1, p_2, \ldots, p_n)$$

If i^{th} component works: $P\{x_i=1\} = p_i$

If i^{th} component fails: $P\{x_i = 0\} = 1 - p_i$

where vector $P = (p_1, p_2, \ldots, p_n)$.

1.SERIES SYSTEM: $\phi(X) = min\ (x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i$

$$R(p) = P\{\phi(X) = 1\} = P\{min\ (x_1, x_2, \ldots, x_n) = 1\} = E[\phi(X)]$$

$$= P\{x_1 = 1, x_2 = 1, \ldots, x_n = 1\} = p_1 p_2 p_3 \ldots p_n$$

$$= \prod_{i=1}^{n} p_i = \prod_{i=1}^{n} E[x_i], \text{ where } E[x_i] = p_i \times 1 + (1 - p_i) \times 0 = p_i$$

Therefore, the reliability function for series systems is:

$$R(p) = E[\phi(X)] = E\left[\prod_{i=1}^{n} x_i\right] = \prod_{i=1}^{n} E[x_i] = \prod_{i=1}^{n} p_i$$

2. PARALLEL SYSTEM: $\phi(X) = max\ (x_1, x_2, \ldots, x_n) = 1 - \prod_{i=1}^{n} (1 - x_i)$

$$R(p) = P\{\phi(X) = 1\} = P\{max\ (x_1, x_2, \ldots, x_n) = 1\}$$

$$= 1 - P\{min\ (x_1, x_2, \ldots, x_n) = 0\}$$

$$= 1 - P\{x_1 = 0, x_2 = 0, \ldots, x_n = 0\}, \text{ where components are independent.}$$

$$= 1 - P\{x_1 = 0\} P\{x_2 = 0\} \ldots P\{x_n = 0\}$$

$$= 1 - (1 - p_1)(1 - p_2) \ldots (1 - p_n) = 1 - \prod_{i=1}^{n} (1 - p_i)$$

Therefore, the reliability function for parallel systems is:

$$R(p) = E[\phi(X)] = 1 - \prod_{i=1}^{n} (1 - p_i)$$

4.5RELIABILITY OF DYNAMIC SYSTEMS (TIME CONSIDERATION)

λ and μ are constants in the static systems, but one should consider time variable in the dynamic system, (see sections 4.6 and 4.7 for details).

RELIABILITY FUNCTION R(t): (for non-repair systems)

R(t)=P{S is functioning in [0 , t]} = P{lifetime of system > t}

$= 1 - F(t) = \bar{F}(t)$, (tail function)

where F(t) stands for system's life time distribution, and P{.} stands for probability measure. At time t, the reliability of ith component is $1 - F_i(t) = \bar{F}_i(t)$; therefore, for a non-repair system the reliability function is defined by:

$$R(t) = \bar{F}(t) = R(\bar{F}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t))$$

$$F(t) = 1 - \bar{F}(t) = 1 - R(\bar{F}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t))$$

1. SERIES SYSTEM:

Reliability function: $R(p) = \prod_{i=1}^{n} p_i$ (for static systems)

Reliability function: $R(t) = \prod_{i=1}^{n} \bar{F}_i(t) = \prod_{i=1}^{n} (1 - F_i(t))$ (for dynamic systems)

Distribution function

$$: F(t) = 1 - R(t) = 1 - \bar{F}(t) = 1 - \prod_{i=1}^{n} \bar{F}_i(t) = 1 - \prod_{i=1}^{n} (1 - F_i(t))$$

2. PARALLEL SYSTEM:

Reliability function: $R(p) = 1 - \prod_{i=1}^{n} (1 - p_i)$ (for static systems)

Reliability function: $R(t) = 1 - \prod_{i=1}^{n} (1 - \bar{F}_i(t)) = 1 - \prod_{i=1}^{n} F_i(t)$

(for dynamic systems)

Distribution function: $F(t) = 1 - R(t) = \prod_{i=1}^{n} F_i(t)$

4.6THE CONCEPT OF FAILURE RATE

Assume that the component works at time t, but it will fail after Δt, then this probability density is called *failure rate* $\lambda(t)$:

$$P\{X \le (t+\Delta t)/(X>t)\} = \frac{P\{t<X\le t+\Delta t\}}{P\{X>t\}}$$

$$= \frac{\int_t^{(t+\Delta t)} f(x)\,dx}{\bar{F}(t)} = \frac{f(t)\,\Delta t}{\bar{F}(t)} = \lambda(t)\,\Delta t.$$

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)} = \frac{-\frac{d}{dt}\bar{F}(t)}{\bar{F}(t)} = \frac{-\frac{d}{dt}R(t)}{R(t)} = \frac{f(t)}{R(t)}$$

Therefore, failure rate $\lambda(t)$ represents the probability intensity that a t-year-old component will fail.

From the distribution function, one can find the failure rate:

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)}, \quad \frac{d}{dt}F(t) = f(t), \qquad \bar{F}(t) = 1-F(t)$$

From the failure rate, one can find the distribution function:

$$F(t) = 1-e^{-\int_0^t \lambda(u)\,du}, \quad \bar{F}(t) = e^{-\int_0^t \lambda(u)\,du}$$

HOW TO CALCULATE THE FAILURE RATE OF A RELIABILITY SYSTEM:

$$\bar{F}(t) = R(\bar{F}_1(t), \ldots, \bar{F}_n(t))$$

$$F(t) = 1-\bar{F}(t) = 1-R(\bar{F}_1(t), \bar{F}_2(t), \ldots, \bar{F}_n(t))$$

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)} = \frac{\frac{d}{dt}F(t)}{\bar{F}(t)} = \frac{-\frac{d}{dt}\bar{F}(t)}{\bar{F}(t)} = \frac{-\frac{d}{dt}R(\bar{F}_1(t), \ldots, \bar{F}_n(t))}{R(\bar{F}_1(t), \ldots, \bar{F}_n(t))}$$

THE FAILURE RATE OF SERIES SYSTEMS:

Assume that the ith component's failure rate is $\lambda_i(t)$.

$$F(t) = 1 - \bar{F}(t) = 1 - R(\bar{F}_1(t), \ldots, \bar{F}_n(t)) = 1 - \prod_{i=1}^{n} \bar{F}_i(t)$$

$$= 1 - \prod_{i=1}^{n} e^{-\int_0^t \lambda_i(t)\,dt} = 1 - e^{-\sum_{i=1}^{n} \int_0^t \lambda_i(t)\,dt} = 1 - e^{-\int_0^t \sum_{i=1}^{n} \lambda_i(t)\,dt}$$

$$\bar{F}(t) = 1 - F(t) = e^{-\int_0^t \sum_{i=1}^{n} \lambda_i(t)\,dt}$$

$$f(t) = \frac{d}{dt}F(t) = e^{-\int_0^t \sum_{i=1}^{n} \lambda_i(t)\,dt} \sum_{i=1}^{n} \lambda_i(t)$$

$$\lambda(t) = \frac{f(t)}{\bar{F}(t)} = \sum_{i=1}^{n} \lambda_i(t)$$

Therefore, the failure rate of a series system is equal to the sum of each component's failure rate.

THE FAILURE RATE OF PARALLEL SYSTEMS:

$$F(t) = 1-\bar{F}(t) = 1-\left(1-\prod_{i=1}^{n}(1-\bar{F}_i(t))\right) = 1-\left(1-\prod_{i=1}^{n}F_i(t)\right)$$

$$= \prod_{i=1}^{n}F_i(t) = \prod_{i=1}^{n}(1-\bar{F}_i(t))$$

$$F(t) = \prod_{i=1}^{n}(1-\bar{F}_i(t)) = \prod_{i=1}^{n}\left(1-e^{-\int_0^t \lambda_i(u)\,dt}\right)$$

From formula:

$$\prod_{i=1}^{n}(1-a_i) = 1-\sum_{i=1}^{n}a_i+\sum_{i\neq j}a_ia_j-\sum_{i\neq j\neq k}a_ia_ja_k+\ldots+(-1)^n a_1a_2\ldots a_n$$

,

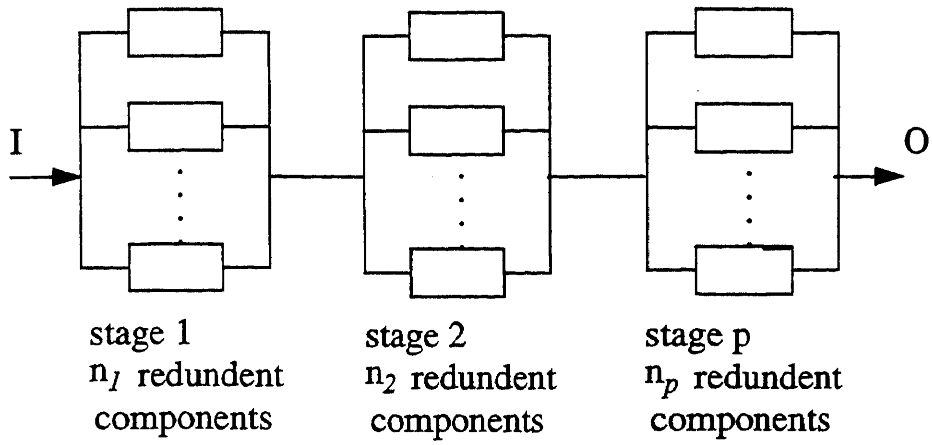
one can obtain: $R(t) = \bar{F}(t) = 1-\prod_{i=1}^{n}\left(1-e^{-\int_0^t \lambda_i(u)\,du}\right)$

$$= \sum_{i=1}^{n}e^{-\int_0^t \lambda_i(u)\,du}-\sum_{i\neq j}e^{-\int_0^t (\lambda_i(u)+\lambda_j(u))\,du}+\ldots+(-1)^n e^{-\int_0^t \sum_{i=1}^{n}\lambda_i(u)\,du}$$

Since $R(t) = \bar{F}(t)$, $\lambda(t)$ is given by:

$$\lambda(t) = \frac{-\frac{d}{dt}\bar{F}(t)}{\bar{F}(t)} = \frac{-\frac{d}{dt}\bar{F}(t)}{R(t)} = \frac{-\frac{d}{dt}R(t)}{R(t)}$$

THE RELIABILITY FUNCTION OF A SERIES-PARALLEL SYSTEM:

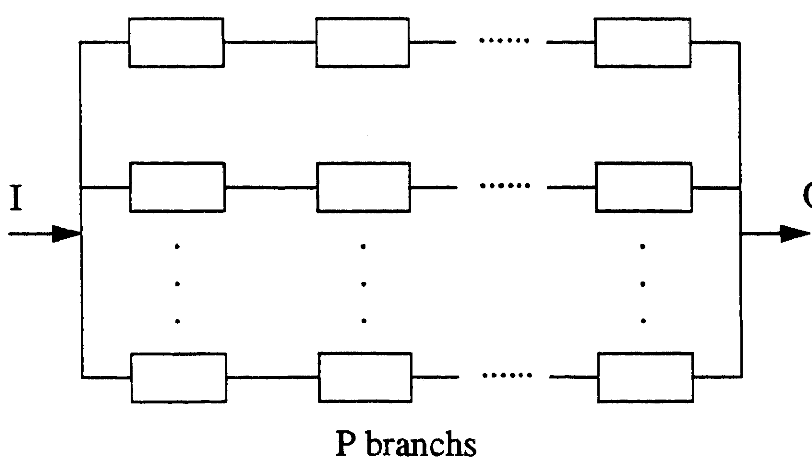


The reliability function for a series-parallel system is given by:

$$R(t) = \bar{F}(t) = \prod_{i=1}^{p}\left(1 - \prod_{j=1}^{n_i}(1 - \bar{F}_{ij}(t))\right) = \prod_{i=1}^{p}\left(1 - \prod_{j=1}^{n_i} F_{ij}(t)\right)$$

where $F_{ij}(t)$: at i^{th} stage, the j^{th} node's distribution function.

THE RELIABILITY FUNCTION OF A PARALLEL-SERIES SYSTEM:



The reliability function for a parallel-series system is given by:

$$R(t) = 1 - \prod_{i=1}^{p}\left(1 - \prod_{j=1}^{n_i} \bar{F}_{ij}(t)\right)$$

where $F_{ij}(t)$: in ith branch, the jth component's distribution function, and each ith branch has n_i components in series.

4.7SYSTEM AVAILABILITY FUNCTION A(t)

When the system is repairable, assume that the system has failed then the repair starts immediately. If the distribution function of repair time is G(t), the repair rate $\mu(t)$ can be defined by:

$$\mu(t) = \frac{\frac{d}{dt}G(t)}{\bar{G}(t)}$$

$$MTTR = \int_0^\infty [1 - G(t)]\,dt; MTTF = \int_0^\infty [R(t)]\,dt = \int_0^\infty [1 - F(t)]\,dt$$

In the case, where the repair rate $\mu(t)$ is a constant μ, it can be proved that:

$$G(t) = 1 - e^{\mu t} \quad \text{and} \quad MTTR = \frac{1}{\mu} = \tau$$

If component i has a failure rate λ_i and a repair rate μ_i, it can be proved that:

$$q_i(t) = \frac{\mu_i}{\lambda_i + \mu_i} + \frac{\lambda_i}{\lambda_i + \mu_i} e^{-(\lambda_i + \mu_i)t}, \text{ and}$$

$$A(t) = R(q_1(t), q_2(t), \ldots, q_n(t))$$

$$= R\left(\frac{\mu_1}{\lambda_i+\mu_1} + \frac{\lambda_1}{\lambda_1+\mu_1}e^{-(\lambda_1+\mu_1)t}, \ldots, \frac{\mu_n}{\lambda_n+\mu_n} + \frac{\lambda_n}{\lambda_n+\mu_n}e^{-(\lambda_n+\mu_n)t}\right)$$

where $q_i(t)$ is the availability function of component i at time t, and $A(t)$ is the availability function of system S at time t.

If $t \to \infty$ (long run), like a static system, the stationary availability function can be given by: $A(\infty) = R\left(\frac{\mu_1}{\lambda_1+\mu_1}, \ldots, \frac{\mu_n}{\lambda_n+\mu_n}\right)$

SYSTEM AVAILABILITY FUNCTION FOR SERIES SYSTEMS

1. *transient:* $A(t) = R(q_1(t), \ldots, q_n(t)) = \prod_{i=1}^{n} q_i(t)$

$$= \prod_{i=1}^{n}\left(\frac{\mu_i}{\lambda_i+\mu_i} + \frac{\lambda_i}{\lambda_i+\mu_i}e^{-(\lambda_i+\mu_i)t}\right)$$

2. *stationary:* $A(\infty) = \prod_{i=1}^{n} q_i(\infty) = \prod_{i=1}^{n}\left(\frac{\mu_i}{\lambda_i+\mu_i}\right)$

SYSTEM AVAILABILITY FUNCTION FOR PARALLEL SYSTEMS

1. *transient:* $A(t) = R(q_1(t), \ldots, q_n(t)) = 1 - \prod_{i=1}^{n}(1 - q_i(t))$

$$= 1 - \prod_{i=1}^{n} \left(1 - \frac{\mu_i}{\lambda_i + \mu_i} - \frac{\lambda_i}{\lambda_i + \mu_i} e^{-(\lambda_i + \mu_i) t} \right)$$

$$= 1 - \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \mu_i} (1 - e^{-(\lambda_i + \mu_i) t})$$

2. *stationary:* $A(\infty) = 1 - \prod_{i=1}^{n} \frac{\lambda_i}{\lambda_i + \mu_i}$

MINIMAL PATH METHOD FOR SYSTEM AVAILABILITY FUNCTIONS

The following notations are defined as:

p_i (t): the probability of the i^{th} minimal path which is functioning at time t.

n components with availability function $q_i(t)$, i = 1,...,n.

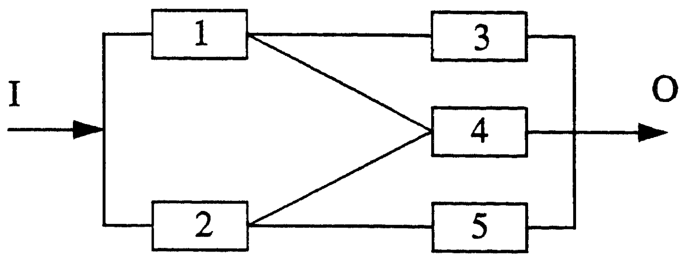
m possible minimal paths: $p_1,...,p_m$.

$$A(t) = P\{\bigcup_{i=1}^{m} p_i(t)\} = \sum_{i=1}^{m} P(p_i(t)) - \sum_{i<j} P(p_i(t) p_j(t)) \ldots\ldots$$

$$+ (-1)^{m-1} P(p_1(t), \ldots, p_m(t))$$

where $P(p_j(t)) = \prod_{i \in p_j} q_i(t)$, (series in the minimal path).

Example 5. Build up the availability function by the minimal path method, and the system is shown below:



The minimal paths are: (1,3),(1,4),(2,4),(2,5).

$$A(t) = P\{\bigcup_{i=1}^{4} p_i(t)\} = [q_1(t)q_3(t) + q_1(t)q_4(t) + q_2(t)q_4(t)]$$

$$+ [q_2(t)q_5(t)] - [q_1(t)q_3(t)q_4(t)] - [q_1(t)q_2(t)q_3(t)q_4(t)]$$

$$-[q_1(t)q_2(t)q_3(t)q_5(t) + q_1(t)q_2(t)q_4(t)]$$

$$-[q_1(t)q_2(t)q_4(t)q_5(t) + q_2(t)q_4(t)q_5(t)]$$

$$+ [q_1(t)q_2(t)q_3(t)q_4(t) + q_1(t)q_2(t)q_3(t)q_4(t)q_5(t)]$$

$$+ [q_1(t)q_2(t)q_3(t)q_4(t)q_5(t) + q_1(t)q_2(t)q_4(t)q_5(t)]$$

$$-[q_1(t)q_2(t)q_3(t)q_4(t)q_5(t)]$$

MINIMAL CUT SET METHOD FOR SYSTEM AVAILABILITY FUNCTIONS

$\bar{A}(t) = 1 - A(t) = P$ { system is not functioning at time t }.

Assume that a system has m minimal cut sets $c_1, \ldots c_m$

$$\bar{A}(t) = \{\bigcup_{i=1}^{m} c_i(t)\} = \sum_{i=1}^{m} P(c_i(t)) - \sum_{i<j} P(c_i(t)\,c_j(t))$$

$$+ \sum_{i<j<k} P(c_i(t)\,c_j(t)\,c_k(t)) + \ldots\ldots + (-1)^{m-1} P(c_1(t), \ldots, c_m(t))$$

where $P(c_i(t)) = \prod_{j \in c_i} \bar{q}_j(t) = \prod_{j \in c_i} (1 - q_j(t))$.

Example 6. Build up the availability function by using the minimal cut set method, and the system is the same as the system in example 5.

The minimal cut sets are: (1,2),(1,4,5),(2,3,4),(3,4,5).

$$\bar{A}(t) = \{\bigcup_{i=1}^{4} c_i(t)\} = (\bar{q}_1\bar{q}_2 + \bar{q}_1\bar{q}_4\bar{q}_5 + \bar{q}_2\bar{q}_3\bar{q}_4 + \bar{q}_3\bar{q}_4\bar{q}_5) -$$

$$(\bar{q}_1\bar{q}_2\bar{q}_4\bar{q}_5 + \bar{q}_1\bar{q}_2\bar{q}_3\bar{q}_4 + \bar{q}_1\bar{q}_2\bar{q}_3\bar{q}_4\bar{q}_5 + \bar{q}_1\bar{q}_2\bar{q}_3\bar{q}_4\bar{q}_5) -$$

$$(\bar{q}_1\bar{q}_3\bar{q}_4\bar{q}_5 + \bar{q}_2\bar{q}_3\bar{q}_4\bar{q}_5) + (\bar{q}_1\bar{q}_2\bar{q}_3\bar{q}_4\bar{q}_5 + \bar{q}_1\bar{q}_2\bar{q}_3\bar{q}_4\bar{q}_5 + \bar{q}_1\bar{q}_2\bar{q}_3\bar{q}_4\bar{q}_5) +$$

$(\bar{q}_1\bar{q}_2\bar{q}_3\bar{q}_4\bar{q}_5) - (\bar{q}_1\bar{q}_2\bar{q}_3\bar{q}_4\bar{q}_5)$. where $\bar{q}_i$ stands for $\bar{q}_i(t)$,and i = 1,2,3,4,5.

4.8DECOMPOSITION OF STRUCTURE FUNCTIONS

Any structure function of order n (n components) can be written as a linear combination of two structure functions of order n-1:

$$\phi(X) = \phi(x_1, x_2, \ldots, x_n) = x_1\phi(1, x_2, \ldots, x_n) + (1-x_1)\phi(0, x_2, \ldots, x_n)$$

$$= [x_1 x_2 \phi(1, 1, x_3, \ldots, x_n) + x_1(1-x_2)\phi(1, 0, x_3, \ldots x_n)]$$

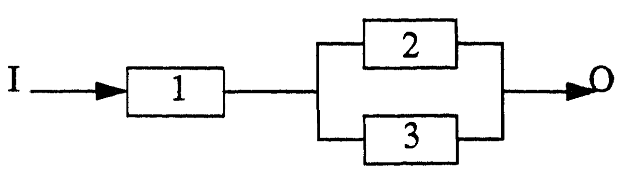
$$+ [(1-x_1)x_2\phi(0, 1, x_3, \ldots, x_n) + (1-x_1)(1-x_2)\phi(0, 0, x_3, \ldots, x_n)]$$

DECOMPOSITION FORMULA

$$\phi(X) = \phi(x_1, x_2, \ldots, x_n) = \sum_{Y \in A} \prod_{j=1}^{n} x_j^{y_j}(1-x_j)^{1-y_j}$$

where $y_i = \{\begin{smallmatrix}1\\0\end{smallmatrix}$, and A is the set of state a such that $\phi(a) = 1$.

Example 7. Build up a structure function by using the decomposition formula, and the system is shown below:



$A = \{(1,1,0), (1,0,1), (1,1,1)\} =$

$\{(y_1, y_2, y_3), (y_1, y_2, y_3), (y_1, y_2, y_3)\}$. where $\phi(a) = 1$

$$\phi(X) = \phi(x_1, x_2, x_3) = \sum_{Y \in A} \left[x_1^{y_1}(1-x_1)^{1-y_1}\right]\left[x_2^{y_2}(1-x_2)^{1-y_2}\right] \times$$

$$\left[x_3^{y_3}(1-x_3)^{1-y_3}\right] = [x_1^1(1-x_1)^0]\,[x_2^1(1-x_2)^0]\,[x_3^0(1-x_3)^1]$$

$$+ [x_1^1(1-x_1)^0]\,[x_2^0(1-x_2)^1]\,[x_3^1(1-x_3)^0]$$

$$+ [x_1^1(1-x_1)^0]\,[x_2^1(1-x_2)^0]\,[x_3^1(1-x_3)^0]$$

$$= x_1x_2(1-x_3) + x_1(1-x_2)x_3 + x_1x_2x_3 = x_1x_2 + x_1x_3 - x_1x_2x_3$$

CHECK: The structure function can also be obtained by the combination of parallel and series systems.

For parallel systems: $\phi(X) = 1 - \prod_{i=1}^{n}(1-x_i)$

For series systems: $\phi(X) = \prod_{i=1}^{n} x_i$

Therefore, $\phi(X) = x_1[1-(1-x_2)(1-x_3)] = x_1x_2 + x_1x_3 - x_1x_2x_3$

4.9THE RELIABILITY IMPORTANCE OF A COMPONENT

From the previous sections, the following equations are given:

$\phi(X) = x_1\phi(1, x_2, \ldots, x_n) + (1-x_1)\phi(0, x_2, \ldots, x_n)$

$R(p) = E[\phi(X)]$, when components are independent,$E[X, Y] = E[X]E[Y]$

$$R(p) = E[\phi(X)] = E[x_1]E[\phi(1, x_2, \ldots, x_n)] + E[(1-x_1)] \times$$
$$E[\phi(0, x_2, \ldots, x_n)] = p_1 R(1, p_2, \ldots, p_n) + (1-p_1) R(0, p_2, \ldots, p_n)$$

The reliability importance of component i is defined as follows:

$$I(i) = \frac{\partial}{\partial p_i} R(p)$$

$$= \frac{\partial}{\partial p_i} [p_i R(p_1, p_2, \ldots, 1, \ldots, p_n) + (1-p_i) R(p_1, \ldots, 0, \ldots, p_n)]$$

$$= R(p_1, p_2, \ldots, 1, \ldots, p_n) - R(p_1, \ldots, 0, \ldots, p_n).$$

Therefore, $\frac{\partial}{\partial p_i} R(p) = R(p_1, p_2, \ldots, 1, \ldots, p_n) - R(p_1, \ldots, 0, \ldots, p_n)$

SERIES SYSTEM: $R(p) = \prod_{i=1}^{n} (1-p_i)$.

$$I(i) = R(p_1, p_2, \ldots, 1, \ldots, p_n) - R(p_1, p_2, \ldots, 0, \ldots, p_n)$$

$$= p_1 p_2 \cdots p_{i-1} \times 1 \times p_{i+1} \cdots p_n - p_1 p_2 \cdots p_{i-1} \times 0 \times p_{i+1} \cdots p_n$$

$= \prod_{j \neq i}^{n} p_j$. Therefore, $I(i) = \prod_{j \neq i}^{n} p_j$

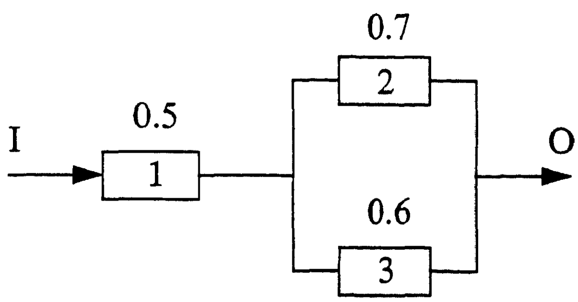
PARALLEL SYSTEM: $R(p) = 1 - \prod_{i=1}^{n} (1-p_i)$

$$I(i) = R(p_1, p_2, \ldots, 1, \ldots, p_n) - R(p_1, p_2, \ldots, 0, \ldots, p_n)$$

$$= \left[1 - \prod_{j \neq i} (1-p_j)(1-1)\right] - \left[1 - \prod_{j \neq i} (1-p_i)(1-0)\right]$$

$$= 1 - 1 + \prod_{j \neq i} (1-p_j) = \prod_{j \neq i}^{n} (1-p_j)\text{. Therefore, } I(i) = \prod_{j \neq i}^{n} (1-p_j)$$

Example 8. A system as shown below has three components, and each component's reliability is given ($p_1 = 0.5, p_2 = 0.7, p_3 = 0.6$). Find the reliability importance for each component.



$$R(p) = p_1[1 - (1-p_2)(1-p_3)] = p_1p_2 + p_1p_3 - p_1p_2p_3$$

$$I(i) = \frac{\partial}{\partial p_i} R(p) = R(p_1, \ldots, 1, \ldots p_n) - R(p_1, \ldots, 0, \ldots p_n)$$

$$I(1) = R(1, 0.7, 0.6) - R(0, 0.7, 0.6)$$

$$= (1 \times 0.7 + 1 \times 0.6 - 1 \times 0.7 \times 0.6) - (0 \times 0.7 + 0 \times 0.6 - 0 \times 0.7 \times 0.6)$$

$$= 1.3 - 0.42 = 0.88$$

$$I(2) = R(0.5, 1, 0.6) - R(0.5, 0, 0.6) = (0.5 \times 1 + 0.5 \times 0.6 - 0.5 \times 1 \times 0.6)$$
$$-(0.5 \times 0 + 0.5 \times 0.6 - 0.5 \times 0 \times 0.6) = 0.5 - 0.3 = 0.2$$

$$I(3) = R(0.5, 0.7, 1) - R(0.5, 0.7, 0) = (0.5 \times 0.7 + 0.5 \times 1 - 0.5 \times 0.7 \times 1)$$

$$-(0.5 \times 0.7 + 0.5 \times 0 - 0.5 \times 0.7 \times 0) = 0.5 - 0.35 = 0.15$$

CHAPTER 5
SOME BASIC THEORIES OF STRUCTURAL RELIABILITY

5.1INTRODUCTION

The reliability of a structure is its ability to perform its design purpose, under some specified conditions, for a reasonably accepted probability of failure.The reliability of a structure is denoted by R and is defined as $R = 1 - P_f$, where P_f is the probability of failure of the structure. Usually, a structure has many possible failure modes; therefore, the first step will usually be to estimate the reliability with respect to each specified failur mode, and then the next step is to estimate the overall reliability of the structure from a system point of view.

With regard to resistance variables it will be assumed that they can be modelled as time-independent random variables, and load variables can only be modelled as stochastic processes. However, in many cases the distribution of the extreme value of a load in the specified period of time can be used.

5.2THE FUNDAMENTAL CASE

In some simple cases the structural reliability is determined by only two independent random variables(a load effect variable S and a resistance variable R) and one failure criterion $R - S \leq 0$. Such a case is called the *fundamental case* and is shown in figure (a). In this fundamental case the probability of failure P_f can be calculated as follows. The probability that the load effect S lies in the interval [x, $x+dx$] is equal to $f_S(x)dx$. Failure will occur if resistance R is smaller than x, and its probability is $F_R(x)$. Therefore, in the interval [x , $x+dx$] the probability of failure is $F_R(x)f_S(x)dx$.

Therefore, the total probability of failure is:

$$P_f = \int_{-\infty}^{\infty} F_R(x) f_S(x)\, dx$$

Figure

(a):

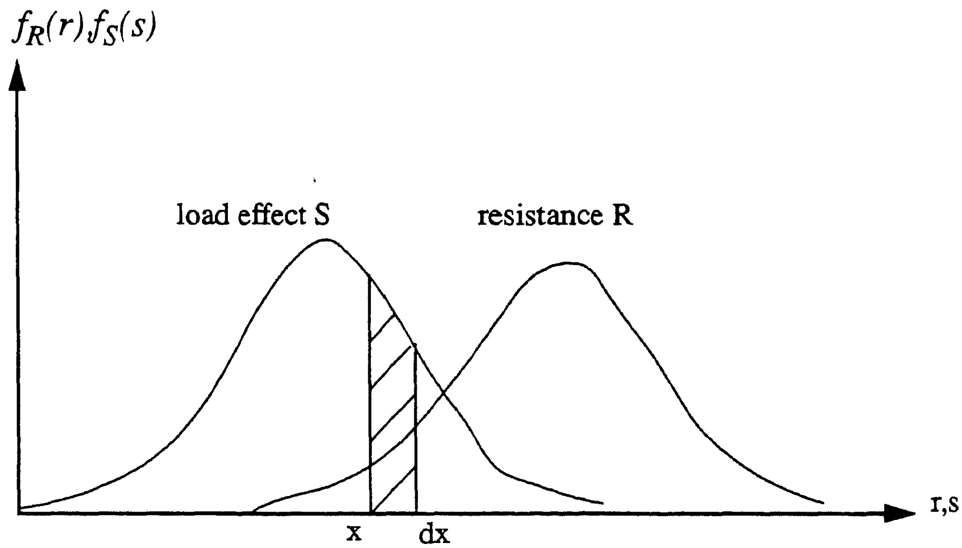
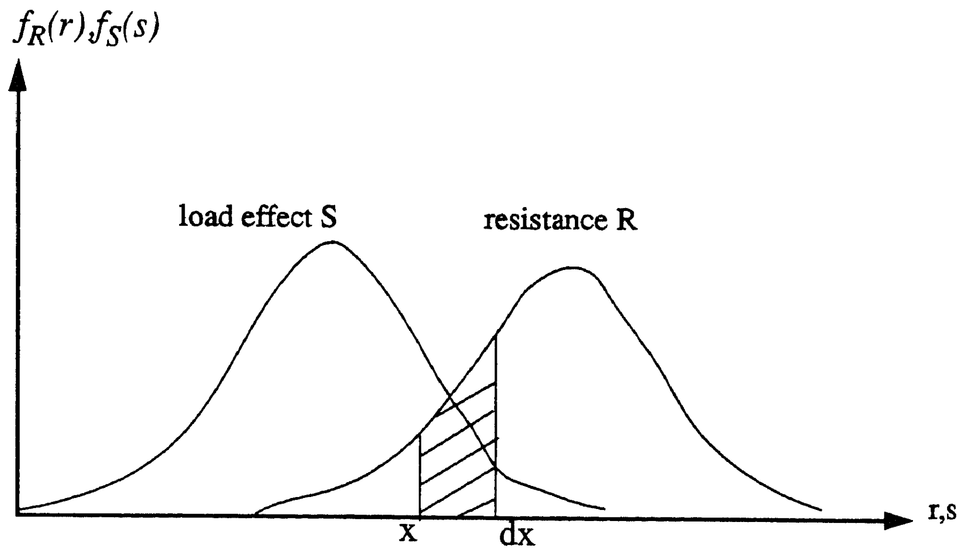


figure (a)

In a similar manner, the probability of failure for the fundamental case can also be written as:

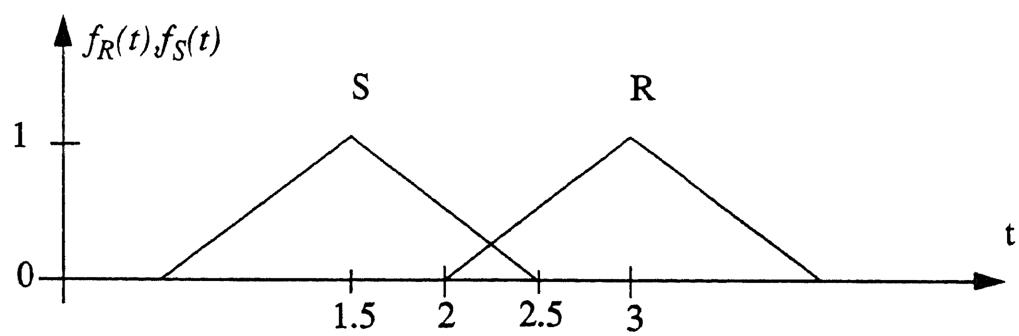
$$P_f = \int_{-\infty}^{\infty} (1 - F_S(x))\, f_R(x)\, dx$$

Figure (b):



Example 1. There are two independent random variables R and S, where R represent

s resistance and S represents load effect, and their distributions are shown below. Find the reliability of this sytem.



$$f_S(t) = a + bt$$

$$f_S(2) = a + 2b = 0.5 \text{-------(1)}$$

$$f_S(2.5) = a + 2.5b = 0 \text{-------(2)}$$

From (2)-(1), $0.5b = -0.5$, b=-1; from (1) a+2 (-1) = 0.5, a=2.5

$$f_S(t) = a + bt = 2.5 - t$$

$$f_R(t) = a + bt$$

$$f_R(2) = a + 2b = 0 \text{-------}(3)$$

$$f_R(2.5) = a + 2.5b = 0.5 \text{-------}(4)$$

From (4)-(3), $0.5\, b = 0.5$, b=1; from (3) a+2 . 1 =0, a=-2

$$f_R(t) = a + bt = -2 + t$$

$$F_R(t) = \int_2^t f_R(x)\, dx = \int_2^t (-2 + x)\, dx = \frac{1}{2}(t^2 - 4t + 4), 2 \le t \le 2.5$$

$$f_S(t) = 2.5 - t = \frac{1}{4}(10 - 4t), 2 \le t \le 2.5$$

Therefore, the probability of failure is given by:

$$P_f = P(R - S \le 0) = \int_2^{2.5} F_R(t) f_S(t)\, dt$$

$$= \int_2^{2.5} \frac{1}{2}(t^2 - 4t + 4) \frac{1}{4}(10 - 4)\, tdt = 0.0026$$

$$R = 1 - P_f = 1 - 0.0026 = 0.9974$$

If R and S are *independent* and *normally distributed*, the probability of failure can be calculated as following:

Let $M = R - S$, then M is also normally distributed, and

$$\mu_M = \mu_R - \mu_S$$

$$\sigma_M^2 = \sigma_R^2 + \sigma_S^2$$

Therefore,

$$P_f = P(R-S\leq 0) = P(M\leq 0) =$$

$$\Phi\left(\frac{0-\mu_M}{\sigma_M}\right) = \Phi\left(\frac{-\mu_M}{\sigma_M}\right) = \Phi\left(-\frac{\mu_R-\mu_S}{\sqrt{\sigma_R^2+\sigma_S^2}}\right) = \Phi(-\beta)$$

where Φ stands for the standard normal distribution function, and $M = R - S$ is called the *safety margin.* μ_M and σ_M are the mean and standard deviation of M.

For the fundamental case the *reliability index* β is defined by:

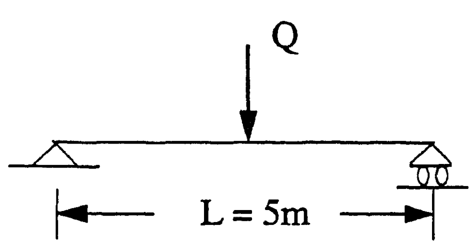
$$\beta = \frac{\mu_M}{\sigma_M}$$

Since, $P_f = \Phi(-\beta)$, $\beta = -\Phi^{-1}(P_f)$; Hence, the one-and-one relationship between the probability of failure P_f and the reliability index β is proven.

Example 2. GIVEN: A simple supported beam loaded as shown below, and

$$\mu_Q = 3 \text{ KN}, \sigma_Q^2 = 1 \text{ KN}^2, \mu_R = 10 \text{ KNm}, \sigma_R^2 = 2.25 \text{ KN}^2\text{m}^2.$$

FIND: The failure probability of the structure.



Usually,this beam will fail by the maximum bending moment at the midpoint.

load effect S $= \frac{Q}{2}\frac{L}{2} = \frac{Q}{2}\frac{5}{2} = \frac{5}{4}Q$, and

$$\mu_S = \frac{5}{4}\mu_Q = \frac{5}{4}3 = 3.75KNm$$

$$\sigma_S^2 = \left(\frac{5}{4}\right)^2\sigma_Q^2 = \frac{25}{16}\times 1 = 1.56\ \mathrm{KN^2m^2}$$

$$\mu_M = \mu_R - \mu_S = 10 - 3.75 = 6.25$$

$$\sigma_M^2 = \sigma_R^2 + \sigma_S^2 = 2.25 + 1.56 = 3.81$$

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{6.25}{\sqrt{3.81}} = 3.20{:}\ P_f = \Phi(-3.20) = 7\times 10^{-4}\ \text{(from the TABLE)}$$

If R and S are *correlated* and normally *distributed,* the probability of failure can be calculated as following:

safety margin $M = R - S$

$$\mu_M = \mu_R - \mu_S$$

$$\sigma_M^2 = \sigma_R^2 + \sigma_S^2 - 2\rho\sigma_R\sigma_S$$

$$\beta = \frac{\mu_M}{\sigma_M}$$

$$P_f = \Phi(-\beta)$$

where ρ is the *correlation coefficient,* which is defined by: $\rho = \dfrac{Cov(R, S)}{\sigma_R \sigma_S}$,

and $Cov(R, S)$ is called the *covariance* of *R* and *S*.

5.3THE CONCEPT OF FAILURE SURFACES

In evaluating the structural reliability, the first step is usually to identify the variables by which the reliability of the structure can be described. Typically, these variables include material strengths, geometrical quantities, and external loads; These variables are called *basic variables* and are modelled as random variables or as stochastic processes, but only modelled by random variables are considered here. Therefore, for a given structure each basic variable has a fixed value. A structure usually has a finite number of basic variables.

Assume that all basic variables are *normally distributed* with the *multivariate joint normal density function* $f_{\bar{X}}$ defined by:

$$f_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^{n/2}|\bar{C}|^{1/2}} e^{\left[-\frac{1}{2}\sum_{i,j=1}^{n}(x_i - \mu_i) M_{ij} (x_j - \mu_j)\right]}$$

where $\bar{x} = (x_1, \ldots, x_n)$; It is convenient to consider the variable $\bar{x}$ as a point in an n-dimensional basic variable space ω, $\bar{M} = \bar{C}^{-1}$, where $\bar{C}$ is the *covariance matrix* defined by:

$$\begin{bmatrix} Var[X_1] & Cov[X_1,X_2] & \ldots & Cov[X_1,X_n] \\ Cov[X_2,X_1] & Var[X_2] & \ldots & Cov[X_2,X_n] \\ \ldots & \ldots & \ldots & \ldots \\ Cov[X_n,X_1] & Cov[X_n,X_2] & \ldots & Var[X_n] \end{bmatrix}$$

Sometimes, basic variables can not be obtained as normal distributions. If this happens a transformation from the non-normal distribution to a normal distribution should be made.

For each *failure mode* of a given set of basic variables, it is possible to determine whether the structure is in a failure state or in a safe state. In other words, the basic variable space ω can be divided into two parts called the *failure region* ω_f and the *safe region* ω_s. The separation of these two parts is called the failure surface and is described by a *failure function*:

$$f(\bar{x}) = f(x_1, \ldots x_n) = 0$$

When failure function is positive, this means the structure is in safe region, and when failure function is negative or zero, this means the structure is in failure region, i.e.

$f(\bar{x}) > 0$, when $\bar{x} \in \omega_s$ (safe region).

$f(\bar{x}) \leq 0$, when $\bar{x} \in \omega_f$ (failure region).

Note that the same failure surfaces can be represented by many different failure functions, and this means that the failure of a structure can happen in a number of different ways (modes).

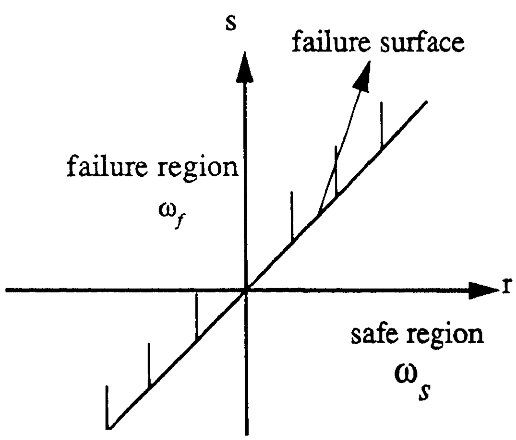
The structural reliability can be calculated by:

$$R = 1 - P_f = 1 - \int_{\omega_f} f_{\bar{X}}(\bar{x})\, d\bar{x}$$

where $f_{\bar{X}}(\bar{x})$ is a joint probability density function with n basic variables and the integral is in n-dimension.

Let $f(\ )$ be a failure function. The safety margin (or failure margin) M can be expressed as $M = f(\bar{x})$; therefore, the *safety margin is not unique* for a given failure surface.

For example, A failure surface described by two random variables is shown below:



The failure function can be expressed by: $f_1(r, s) = r - s$

The corresponding safety margin is: $M_1 = f_1(r, s) = r - s$

The failure function can also be expressed by: $f_2(r, s) = (r - s)^5$

The corresponding safety margin is: $M_2 = f_2(r, s) = (r - s)^5$

When X_i, i = 1, ... , n are *normally distributed* and *uncorrelated* variables with a linear safety margin:

$M = R - S = a_0 + a_1 X_1 + \ldots + a_n X_n$, where a_i, $i = 1, \ldots n$ are constants.

Therefore,

$$\mu_M = a_0 + a_1\mu_{X_1} + \ldots + a_n\mu_{X_n}$$

$$\sigma_M^2 = a_1^2\sigma_{X_1}^2 + \ldots + a_n^2\sigma_{X_n}^2$$

The reliability index β can be used unchanged: $\beta = \dfrac{\mu_M}{\sigma_M}$

When X_i, i = 1, ... , n are *normally distribution* and *correlated* variables with a linear safety margin:

$$M = R - S = a_0 + a_1X_1 + \ldots + a_nX_n$$

Therefore,

$$\mu_M = a_0 + a_1\mu_{X_1} + \ldots + a_n\mu_{X_n}$$

$$\sigma_M^2 = a_1^2\sigma_{X_1}^2 + \ldots + a_n^2\sigma_{X_n}^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} \rho_{X_iX_j}a_ia_j\sigma_{X_i}\sigma_{X_j}, \text{ OR}$$

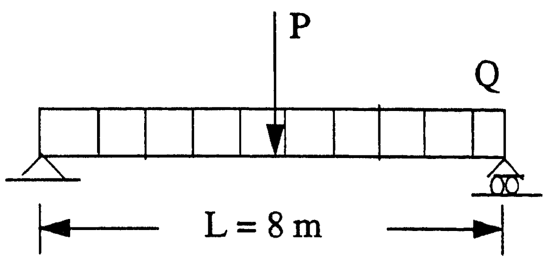
$$\sigma_M^2 = a_1^2\sigma_{X_1}^2 + \ldots + a_n^2\sigma_{X_n}^2 + \sum_{i=1}^{n}\sum_{j=1, j\neq i}^{n} a_ia_jCov\,[X_i, X_j]$$

$$\rho_{X_iX_j} = \frac{Cov\,[X_i, X_j]}{\sigma_{X_i}\sigma_{X_j}}, i \neq j, \text{ and } \beta = \frac{\mu_M}{\sigma_M}.$$

Example 3. . GIVEN: A simply supported beam loaded as shown below. Assume that the beam fails when | M | $\geq M_F$ at midpoint, where M_F is a critical limit moment and M is the bending moment at the midpoint, and the random variables $\overline{X} = (P, Q, M_F)$ with $\mu_{\overline{X}} = (30KN, 20\frac{KN}{m}, 250KNm)$

and $\overline{C} = \begin{bmatrix} 9KN^2 & 3\frac{KN^2}{m} & 0 \\ 3\frac{KN^2}{m} & 6\left(\frac{KN}{m}\right)^2 & 0 \\ 0 & 0 & 50\,(KNm)^2 \end{bmatrix}$

FIND: calculate the probability of failure for the structure



$$|M| = M_1 + M_2 = \frac{1}{4}PL + \frac{1}{8}QL^2$$

$$M = R - S = a_0 + a_1X_1 + a_2X_2 + a_3X_3 = M_F - |M| = M_F - \frac{1}{4}PL - \frac{1}{8}QL^2$$

$$= M_F - \frac{1}{4}8P - \frac{1}{8}8^2Q = -2P - 8Q + M_F$$

Therefore, $a_0 = 0,\ a_1 = -2,\ a_2 = -8,\ a_3 = 1$

$$\mu_M = a_1\mu_{X_1} + a_2\mu_{X_2} + a_3\mu_{X_3} = -2\mu_P - 8\mu_Q - \mu_{M_F} = -2\times 30 - 8\times 20 +$$

$$250 = 30$$

$$\sigma_M^2 = a_1^2\sigma_{X_1}^2 + a_2^2\sigma_{X_2}^2 + a_3^2\sigma_{X_3}^2 + a_1a_2Cov\,[X_1, X_2]$$

$$+ a_1a_3Cov\,[X_1, X_3] + a_2a_1Cov\,[X_2, X_1] + a_2a_3Cov\,[X_2, X_3]$$

$$+ a_3a_1Cov\,[X_3, X_1] + a_3a_2Cov\,[X_3, X_2]$$

$$= 4\times 9 + 64\times 6 + 50 + 16\times 3 + 16\times 3 = 566$$

$$\sigma_M = \sqrt{566} = 23.79$$

$$\beta = \frac{\mu_M}{\sigma_M} = \frac{30}{23.79} = 1.26$$

$$P_f = \Phi(-\beta) = \Phi(-1.26) = 0.1038$$

5.4 THE CONCEPT OF LINEARIZATION AND NORMALIZATION

In practice, it is almost impossible to describe a failure surface by a linear failure function; therefore, if the failure surface is hyperplane, then there exists a linear failure function and it should be used in favior of a non-linear failure function because the probability of failure can easily be calculated for a linear function, but is more complex for a non-linear failure function. However, the choice of linearization point should be considered. Let the *non-linear* safety margin be given by:

$$M = f(\bar{R}) = f(r_1, \ldots, r_n)$$

Expanding M by Taylor series about the linearization point $\bar{r}^0 = (r_1^0, \ldots, r_n^0)$ and keeping only the linear terms:

$$M = f(\bar{R}) \cong f(\bar{r}^0) + \sum_{i=1}^{n} \frac{\partial}{\partial r_i} f(r_i - r_i^0)$$

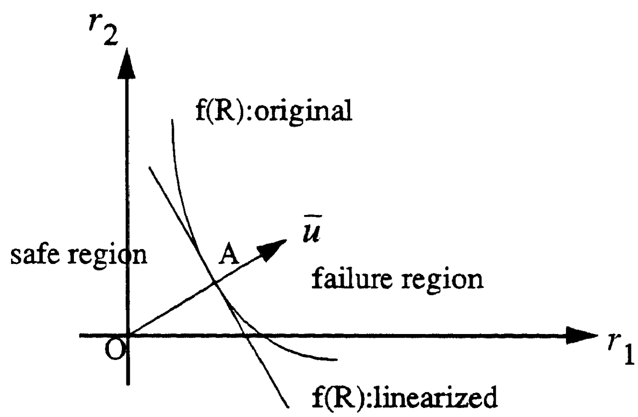
Therefore,the approximation values for μ_M and σ_M are given by:

$$\mu_M \cong f(\bar{r}^0) + \sum_{i=1}^{n} \frac{\partial}{\partial r_i} f(\mu_{r_i} - r_i^0)$$

$$\sigma_M^2 \cong \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial}{\partial r_i} f \frac{\partial}{\partial r_j} f Cov[r_i, r_j]$$

and $\beta = \dfrac{\mu_M}{\sigma_M}$

In a two-dimensional space, a straight line can be expressed by a unit normal vector and a distance from the origin.



$$f(r_1, r_2) = ar_1 + br_2 + c = \frac{\partial f}{\partial r_1} r_1 + \frac{\partial f}{\partial r_2} r_2 + c, \text{ since } a = \frac{\partial f}{\partial r_1}, b = \frac{\partial f}{\partial r_2}.$$

$$= \frac{\frac{\partial f}{\partial r_1}}{\sqrt{\left(\frac{\partial f}{\partial r_1}\right)^2 + \left(\frac{\partial f}{\partial r_2}\right)^2}} r_1 + \frac{\frac{\partial f}{\partial r_2}}{\sqrt{\left(\frac{\partial f}{\partial r_1}\right)^2 + \left(\frac{\partial f}{\partial r_2}\right)^2}} r_2 + \frac{c}{\sqrt{\left(\frac{\partial f}{\partial r_1}\right)^2 + \left(\frac{\partial f}{\partial r_2}\right)^2}}$$

$$= \frac{a}{\sqrt{a^2 + b^2}} r_1 + \frac{b}{\sqrt{a^2 + b^2}} r_2 + \frac{c}{\sqrt{a^2 + b^2}}$$

$$\text{unit vector } \bar{u} = \left(\frac{\frac{\partial f}{\partial r_1}}{\sqrt{\frac{\partial f}{\partial r_1}^2 + \frac{\partial f}{\partial r_2}^2}}, \frac{\frac{\partial f}{\partial r_2}}{\sqrt{\frac{\partial f}{\partial r_1}^2 + \frac{\partial f}{\partial r_2}^2}} \right) \text{or} \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right)$$

$$\text{distance } \overline{OA} = \frac{c}{\sqrt{\frac{\partial f}{\partial r_1}^2 + \frac{\partial f}{\partial r_2}^2}}, \text{ or } \frac{c}{\sqrt{a^2+b^2}}$$

Now consider a fundamental case with two independent variables R and S and the safety margin M = R - S; its mean values are μ_R and μ_S, and the standard deviations are σ_R and σ_S.

By using the normalization formula, one can get:

$$r_1 = \frac{R-\mu_R}{\sigma_R}, R = \sigma_R r_1 + \mu_R, \quad and \quad r_2 = \frac{S-\mu_S}{\sigma_S}, S = \sigma_S r_2 + \mu_S$$

The failure function *f(R,S)* will be transformed into a straight line in the normalized coordinate system (r_1, r_2), and then the failure function $f(r_1, r_2)$ is given by:

$$f(R, S) = R - S = (\sigma_R r_1 + \mu_R) - (\sigma_S r_2 + \mu_S)$$

$$= \sigma_R r_1 - \sigma_S r_2 + (\mu_R - \mu_S) = 0$$

therefore,

$$f(r_1, r_2) = \sigma_R r_1 - \sigma_S r_2 + (\mu_R - \mu_S) = 0$$

Therefore, the shortest distance from the origin can be given by:

$$\frac{\mu_R - \mu_S}{\sqrt{\sigma_R^2 + \sigma_S^2}}, \text{and} \quad = \beta$$

From the geometrical definition, it can be proven that the reliability index β is the shortest distance from the origin to the linear failure surface; therefore, the linearization point must be chosen at the reliability index point. The so-called *Hasofer and Lind reliability index* is defined as the shortest distance from the origin to the failure surface in the normalized coordinate system, and by this definition the reliability index for a non-linear failure surface is equal to the reliability index for the linear tangent hyperplane.

In the general case, a non-linear failure function usually consists of n basic variables; therefore, the calculation of the reliability index must be done by an iterative method:

$$f(\bar{Z}) = f(\beta\bar{U}) \Rightarrow f(z_1, \ldots, z_n) = f(\beta u_1, \ldots, \beta u_n) = 0, \text{ and}$$

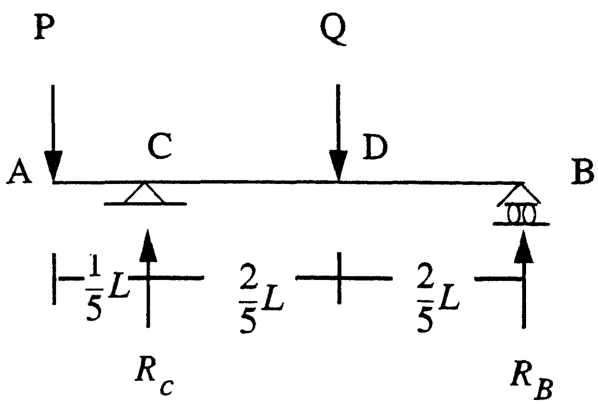
$$u_i = -\frac{1}{g}\frac{\partial}{\partial z_i}(\beta\bar{U}), i = 1, \ldots, n \quad \text{and} \quad g = \sqrt{\sum_{j=1}^{n}\left(\frac{\partial f}{\partial z_j}(\beta\bar{U})\right)^2}$$

Example 4. Consider a simply supported beam loaded as shown below, and assume that the beam will fail when $|M| \geq M_F$, where M_F is the critical bending moment and M is the maximum bending moment. Assume that P and Q are correlated, where $Cov[P, Q] = 0.2 \text{ KN}^2$, and other random variables are uncorrelated to one another with:

$$\mu_P = 2KN, \mu_Q = 15KN, \mu_L = 5m, \mu_{M_F} = 16KNm$$

$$\sigma_P = 0.4KN, \sigma_Q = 3KN, \sigma_L = 0.5m, \sigma_{M_F} = 3KNm$$

Find: the reliability index and its corresponding probability of failure.



$$\sum M_B = 0 \qquad R_C \times \frac{4}{5}L - PL - Q\frac{2}{5}L = 0, \quad R_C = \frac{5}{4}(P + \frac{2}{5}Q)$$

$$\sum F_y = 0 \qquad R_C + R_B = P + Q \quad R_B = P + Q - R_C = \frac{1}{2}Q - \frac{1}{4}P$$

From strength of materials, one can expect that the maximum bending moment M will occur at point D.

$M = R_B \times \frac{2}{5}L = \frac{1}{10}(2Q - P)L = \frac{1}{10}LS$, where $S = 2Q - P$; therefore,

$$\mu_S = 2\mu_Q - \mu_P = 2 \times 15 - 2 = 28$$

$$\sigma_S^2 = 4\sigma_Q^2 + \sigma_P^2 - 2 \times 1 \times 2 \times Cov[P, Q] = 4 \times 9 + 0.16 - 4 \times 0.2 = 35.36$$

,

$$\sigma_S = 5.95$$

The failure function can be given by:

$$f(M_F, L, S) = M_F - \frac{1}{10}LS = 0$$

The random variables M_F, L and S are normalized by

$$Z_1 = \frac{M_F - \mu_{M_F}}{\sigma_{M_F}}, M_F = \mu_{M_F} + \sigma_{M_F} Z_1 = 16 + 3Z_1$$

$$Z_2 = \frac{L - \mu_L}{\sigma_L}, L = \mu_L + \sigma_L Z_2 = 5 + 0.5Z_2$$

$$Z_3 = \frac{S - \mu_S}{\sigma_S}, S = \mu_S + \sigma_S Z_3 = 28 + 5.95Z_3$$

$$M_F - \frac{1}{10} LS = (16 + 3Z_1) - \frac{1}{10}(5 + 0.5Z_2)(28 + 5.95Z_3)$$

$$= 2 + 3Z_1 - 1.4Z_2 - 2.98Z_3 - 0.3Z_2 Z_3$$

$$= 2 + 3\beta u_1 - 1.4\beta u_2 - 2.98\beta u_3 - 0.3\beta\beta u_2 u_3 = 0$$

$$f(\bar{Z}) = f(\beta\bar{U}) = 2 + 3\beta u_1 - 1.4\beta u_2 - 2.98\beta u_3 - 0.3\beta\beta u_2 u_3 = 0$$

$$u_1 = -\frac{1}{g}3, u_2 = \frac{1}{g}(1.4 + 0.3\beta u_3), u_3 = \frac{1}{g}(2.98 + 0.3\beta u_2)$$

$$g = \sqrt{3^2 + (1.4 + 0.3\beta u_3)^2 + (2.98 + 0.3\beta u_2)^2}$$

Therefore, one can obtain:

$$\beta = 0.446, u_1 = -0.66, u_2 = 0.33, u_3 = 0.67$$

$$P_f = \Phi(-\beta) = \Phi(-0.446) = 0.33$$

$$f(\bar{Z}) = \beta + u_1 Z_1 + u_2 Z_2 + u_3 Z_3 = 0.446 - 0.66Z_1 + 0.33Z_2 + 0.67Z_3$$

When the basic variables $\bar{X} = (x_1, \ldots, x_n)$ are *correlated* and the failure surface is *non-linear*:

The first step is: to find the uncorrelated variables $\bar{Y} = (y_1, \ldots, y_n)$ i.e. to find the eigenvalues and eigenvectors.

The second step is: to normalize these uncorrelated variables and to obtain the normalized and uncorrelated variables $\bar{Z} = (z_1, \ldots, z_n)$ by this transformation formula:

$$z_i = \frac{y_i - \mu_{y_i}}{\sigma_{y_i}}, i = 1, \ldots, n$$

The last step is: to find the linear tangent hyperplane to the failure surface of $f(\bar{Z}) = 0$, and then to solve the probability index β value, and $P_f = \Phi(-\beta)$

Assume that the covariance matrix of vector $\bar{X}$ is:

$$\bar{C}_{\bar{X}} = \begin{bmatrix} Var[x_1] & Cov[x_1, x_2] & \ldots & Cov[x_1, x_n] \\ \ldots & \ldots & \ldots & \ldots \\ Cov[x_n, x_1] & Cov[x_n, x_2] & \ldots & Var[x_n] \end{bmatrix}$$

By the linear algebra theorems, the transformation from the correlated variables $\bar{X}$ to the uncorrelated variables $\bar{Y}$ can be obtained by:

$$\bar{Y} = \bar{A}^T\bar{X}$$

where $\bar{A}$ is an orthogonal matrix, and each column of $\bar{A}$ is the orthonormal eigenvectors of $\bar{C}_{\bar{X}}$, and $\bar{A}^T = A^{-1}$.

Then, the uncorrelated diagonal matrix $\overline{C}_{\overline{Y}}$ can be obtained by:

$$\overline{C}_{\overline{Y}} = \overline{A}^T \overline{C}_{\overline{X}} \overline{A} = \begin{bmatrix} Var\,[y_1] & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & Var\,[y_n] \end{bmatrix}$$

Each element in the uncorrelated diagonal matrix $\overline{C}_{\overline{Y}}$ is the eigenvalue of correlated matrix $\overline{C}_{\overline{X}}$ i.e. $Var\,[y_i]$, i=1, ... ,n, are equal to the eigenvalues of $\overline{C}_{\overline{X}}$

From $\overline{Y} = \overline{A}^T \overline{X}$, one can get:

$$E\,[\overline{Y}] = \overline{A}^T E\,[\overline{X}] \text{ or } \begin{bmatrix} E[y_1] \\ \dots \\ E[y_n] \end{bmatrix} = \overline{A}^T \begin{bmatrix} E\,[x_1] \\ \dots \\ E\,[x_n] \end{bmatrix}$$

The transformation form, $z_i = \dfrac{y_i - \mu_{y_i}}{\sigma_{y_i}}$, $i = 1, \dots, n$ can be written as the vectorial

form: $\overline{Z} = \overline{C}_{\overline{Y}}^{-1/2} (\overline{Y} - E\,[\overline{Y}])$, where $\overline{C}_{\overline{Y}} = \overline{A}^T \overline{C}_{\overline{X}} \overline{A}$, $\overline{Y} = \overline{A}^T \overline{X}$,and

$E\,[\overline{Y}] = \overline{A}^T E\,[\overline{X}]$

Therefore, the transformation from the correlated variables $\overline{X}$ to the uncorrelated and normalized variables $\overline{Z}$ can be given by:

$$\bar{Z} = (\bar{A}^T \bar{C}_{\bar{X}} \bar{A})^{-1/2} (\bar{A}^T \bar{X} - \bar{A}^T E[\bar{X}])$$

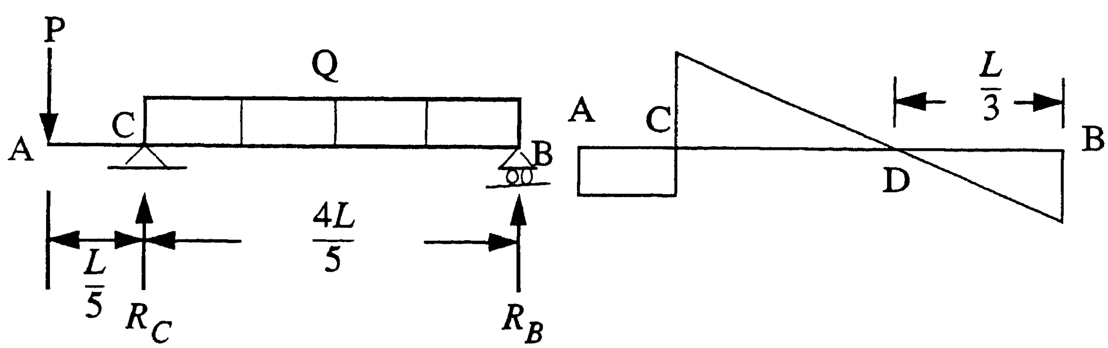
$$= (\bar{A}^T \bar{C}_{\bar{X}} \bar{A})^{-1/2} \bar{A}^T (\bar{X} - E[\bar{X}]).$$

Example 5. Consider a simply supported beam loaded as shown below, and assume that the beam will fail when $|M| \geq M_F$, where M_F is the critical bending moment and M is the maximum bending moment. Let the basic variables $\bar{X} = (P, Q, L, M_F)$ be given by the mean vector:

$$E[\bar{X}] = \left[\mu_P, \mu_Q, \mu_L, \mu_{M_F}\right] = \left[2KN, 1.5\frac{KN}{m}, 5m, 3KNm\right]$$

and by the covariance matrix:

$$\bar{C}_{\bar{X}} = \begin{bmatrix} 0.16 & 0.1 & 0 & 0 \\ 0.1 & 0.09 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.36 \end{bmatrix}$$



From strength of materials, one can find that the maximum bending moment will occur at D, where $length = \frac{L}{3}$ from the right edge.

$$\sum M_B = 0; \quad R_C \times 4 - P \times 5 - 4Q \times 2 = 0,$$

$$4R_C = 5P + 8Q, R_C = \frac{5}{4}P + 2Q$$

$$\sum F_y = 0; \quad R_B = P + 4Q - R_C = 2Q - \frac{1}{4}P$$

$$|M| = \frac{1}{2} \times \frac{L}{3} \times R_B = \frac{1}{6}L\,(2Q - \frac{1}{4}P) = \frac{1}{24}L\,(8Q - P)$$

Only P and Q are correlated; therefore, eigenvalues are calculated for the matrix:

$$\bar{C} = \begin{bmatrix} 0.16 & 0.1 \\ 0.1 & 0.09 \end{bmatrix}$$

$$\left| \begin{bmatrix} 0.16 - \lambda & 0.1 \\ 0.1 & 0.09 - \lambda \end{bmatrix} \right| = 0 \; ; \; (0.16 - \lambda)\;(0.09 - \lambda) - 0.1 \times 0.1 = 0$$

$$\lambda_1 = 0.01905, \lambda_2 = 0.2309$$

$$\begin{bmatrix} 0.16 - \lambda & 0.1 \\ 0.1 & 0.09 - \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for $\lambda_1 = 0.01905$, it gives the corresponding orthonormal eigenvector:

$$\begin{bmatrix} 0.14095 & 0.1 \\ 0.1 & 0.07095 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad 0.14095x + 0.1y = 0; \qquad y = -1.4095x$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1.4095 \end{bmatrix} \Rightarrow \begin{bmatrix} \dfrac{1}{\sqrt{1^2 + (-1.4095)^2}} \\ -\dfrac{-1.4095}{\sqrt{1^2 + (-1.4095)^2}} \end{bmatrix} = \begin{bmatrix} 0.5786 \\ -0.8156 \end{bmatrix}$$

for $\lambda_2 = 0.2309$, it gives the corresponding orthonormal eigenvector:

$$\begin{bmatrix} -0.0709 & 0.1 \\ 0.1 & -0.1409 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; -0.0709x + 0.1y = 0, \quad y = 0.709x$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0.709 \end{bmatrix} \Rightarrow \begin{bmatrix} \dfrac{1}{\sqrt{1^2 + 0.709^2}} \\ \dfrac{0.709}{\sqrt{1^2 + 0.709^2}} \end{bmatrix} = \begin{bmatrix} 0.8158 \\ 0.5784 \end{bmatrix}$$

Therefore, the transformation matrix $\bar{A}$ is given by:

$$\bar{A} = \begin{bmatrix} 0.5786 & 0.8158 \\ -0.8156 & 0.5784 \end{bmatrix}$$

and the uncorrelated variables Y_1 and Y_2 are given by $\bar{Y} = \bar{A}^T \bar{X}$:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0.5786 & -0.8156 \\ 0.8158 & 0.5784 \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0.5786P - 0.8156Q \\ 0.8158P + 0.5784Q \end{bmatrix}$$

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0.5786 & -0.8156 \\ 0.8158 & 0.5784 \end{bmatrix}^{-1} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \begin{bmatrix} 0.5786 & 0.8158 \\ -0.8156 & 0.5784 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

$$\begin{bmatrix} P \\ Q \end{bmatrix} = \begin{bmatrix} 0.5786Y_1 + 0.8158Y_2 \\ -0.8156Y_1 + 0.5784Y_2 \end{bmatrix}$$

Therefore, the uncorrelated variables are

$$\bar{Y} = [Y_1, Y_2, Y_3, Y_4] = [(0.579P - 0.816Q), (0.8158P + 0.5784Q), L, M_F]$$

The safety margin

$$M = M_F - |M| = M_F - \frac{1}{24}L(8Q - P) = M_F - \frac{1}{3}LQ + \frac{1}{24}Lp = 0$$

can be written in uncorrelated variables by

$$M = Y_4 - \frac{1}{3}Y_3(-0.8156Y_1 + 0.5784Y_2) + \frac{1}{24}Y_3(0.5786Y_1 + 0.8158Y_2) = 0$$

$$M = Y_4 + 0.296Y_1Y_3 - 0.159Y_2Y_3 = 0$$

$$\mu_{Y_1} = E[Y_1] = 0.5786\mu_P - 0.8156\mu_Q = 0.5338$$

$$\sigma_{Y_1}^2 = 0.5786^2\sigma_P^2 + 0.8156^2\sigma_Q^2 - 2 \times 0.5786 \times 0.8156 \times 0.1 = 0.0191$$

$$\sigma_{Y_1} = 0.138$$

$$\mu_{Y_2} = E[Y_2] = 0.8158\mu_P + 0.5784\mu_Q = 2.4992$$

$$\sigma_{Y_2}^2 = 0.8158^2\sigma_P^2 + 0.5784^2\sigma_Q^2 + 2 \times 0.8158 \times 0.5784 \times 0.1 = 0.231$$

$$\sigma_{Y_2} = 0.4806$$

$$Z_1 = \frac{Y_1 - \mu_{Y_1}}{\sigma_{Y_1}} = \frac{Y_1 - 0.5338}{0.138}, \quad Y_1 = 0.138Z_1 + 0.5338$$

$$Z_2 = \frac{Y_2 - \mu_{Y_2}}{\sigma_{Y_2}} = \frac{Y_2 - 2.4992}{0.4806}, \quad Y_2 = 0.4806Z_2 + 2.4992$$

$$Z_3 = \frac{Y_3 - \mu_{Y_3}}{\sigma_{Y_3}} = \frac{Y_3 - 5}{0.25}, \quad Y_3 = 0.25Z_3 + 5$$

$$Z_4 = \frac{Y_4 - \mu_{Y_4}}{\sigma_{Y_4}} = \frac{Y_4 - 3}{0.36}, \quad Y_4 = 0.36Z_4 + 3$$

$$M = Y_4 + 0.296Y_1Y_3 - 0.159Y_2Y_3$$

$$= (0.36Z_4 + 3) + 0.296\,(0.138Z_1 + 0.5338)\ (0.25Z_3 + 5)$$

$$-0.159\,(0.4806Z_2 + 2.4992)\ (0.25Z_3 + 5) = 0$$

$$M = 1.8031 + 0.2024Z_1 - 0.3821Z_2 - 0.0598Z_3 + 0.36Z_4 + 0.0102Z_1Z_3$$
$$-0.0191Z_2Z_3 = 0$$

$$f(\bar{Z}) =$$
$$1.8031 + 0.2024Z_1 - 0.3821Z_2 - 0.0598Z_3 + 0.36Z_4 + 0.0102Z_1Z_3$$
$$-0.0191Z_2Z_3 = 0$$

$$f(\bar{Z}) = f(\beta\bar{U}) = 1.8031 + 0.2042\beta u_1 - 0.3821\beta u_2 - 0.0598\beta u_3$$

$$+0.36\beta u_4 + 0.0102\beta^2 u_1 u_3 - 0.0191\beta^2 u_2 u_3 = 0$$

$$\beta = \frac{1.8031}{-0.2042u_1 + 0.3821u_2 + 0.0598u_3 - 0.36u_4 - 0.01\beta u_1 u_3 + 0.0191\beta u_2 u_3}$$

$$u_1 = -\frac{1}{g}0.2042;\ u_2 = \frac{1}{g}(0.3821 + 0.0191\beta u_3)$$

$$u_3 = \frac{1}{g}(0.0598 - 0.0102\beta u_1 + 0.0191\beta u_2)\ ;\ u_4 = -\frac{1}{g}0.36$$

$$g = \sqrt{0.204^2 + (0.382 + 0.019\beta u_3)^2 + (0.06 - 0.01\beta u_1 + 0.019\beta u_2)^2 + 0.36^2}$$

Therefore, one can obtain:

$$\beta = 3.14,\ u_1 = -0.352,\ u_2 = 0.676,\ u_3 = 0.196,\ u_4 = -0.621$$

$$P_f = \Phi(-\beta) = \Phi(-3.14) = 0.0008447$$

$$f(\bar{Z}) = \beta + u_1 Z_1 + u_2 Z_2 + u_3 Z_3 + u_4 Z_4$$

$$f(\bar{Z}) = 3.41 - 0.352Z_1 + 0.676Z_2 + 0.196Z_3 - 0.621Z_4$$

CHAPTER 6
MODELLING OF STRUCTURES

6.1INTRODUCTION

A real structure is so complex that a complete calculation of the failure probability is impossible. Usually, there are a large number of different failure modes so that they cannot all be taken into account; therefore, it is necessary to idealize the structure so that the calculation of the probability of failure becomes manageable, and to build up the model carefully so that the most important failure modes are chosen to reflect the real structure closely. It is assumed that the reliability of a structure is estimated on the basis of a series system modelling, where the components are failure modes, and the failure modes are modelled by parallel systems.

The modelling used in this thesis is based on the assumption that the probability of failure of a structure can be sufficiently accurately estimated by choosing only a finite number of significant failure modes and then by combining them in a complex system. One of the main problems in the structural reliability analysis is to identify the significant failure modes; Several methods to identify the failure modes have been suggested in the last decade, In this thesis Using ANSYS finite element software to accurately identify the failure modes is described in detail. In this thesis only truss structures are considered, but the method used can be extended easily.

6.2MODELLING OF FUNDAMENTAL STRUCTURAL SYSTEMS

Two fundamental systems, *series systems* and *parallel systems*, will be discussed in this section, and these two fundamental systems occupy the most important parts in the modelling of structural systems.

HOW TO DECIDE A TRUSS STRUCTURE IS STATICALLY DETERMINATE OR STATICALLY INDETERMINATE:

Let b stand for the number of bars, j stand for the number of joints, r stand for the number of reaction components, and e stand for the number of incomplete equilibrium equations. Therefore, for a two-dimensional structure, one may defind:

for a statically determinate structure, $b + r = 2j - e$

for a statically indeterminate structure, $b + r > 2j - e$

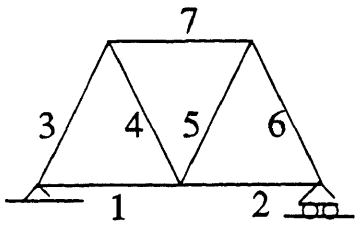
for a mechanism, $b + r < 2j - e$

Since there are $b + r$ unknowns, and $2j - e$ equilibrium equations.

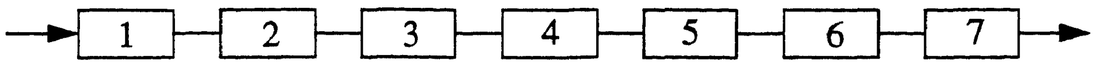
In three-dimensional structures. The definition for statical determinacy becomes :

$b + r = 3j - e$

Consider a statically determinate structure with seven bars as shown below. The total number of failure modes is also seven, because for a statically determinate structure the whole structure fails as soon as any structural member fails. This can be symbolized by the series system as shown below:



b + r = 2 j - e ; 7 + 3 = 2 . 5 - 0
10 = 10 ; statically determinate



Let the random variable R be the strength of a series system, and let the random variable R_i be the strength of member i, where $i = 1, \dots, n$. Let a random load S with the density function f_S be loaded on the series system, and result in a load effect S_i in menber i. Let F_{R_i} be the distribution function for the variable R_i, then the distribution function F_R for the total series system is given by:

$$F_R(S) = P(R \leq S) = 1 - P(R > S)$$

$$= 1 - P[(R_1 > S_1) \cap (R_2 > S_2) \cap \dots \cap (R_n > S_n)]$$

$$= 1 - (1 - F_{R_1}(S_1))(1 - F_{R_2}(S_2)) \dots (1 - F_{R_n}(S_n))$$

$$= 1 - \prod_{i=1}^{n} (1 - F_{R_i}(S_i))$$

where it is assumed that R_i are independent.

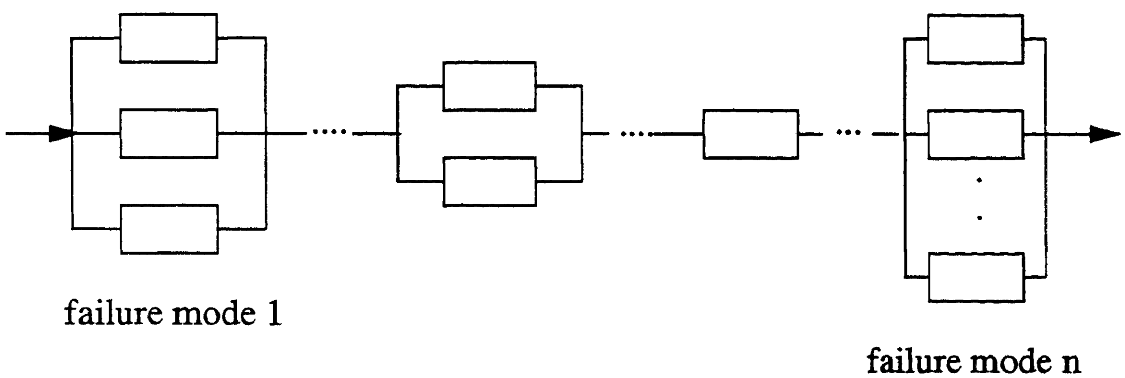
Then the probability of failure P_f for the series system is given by:

$$P_f = \int_{-\infty}^{\infty} F_R(S) f_S(S)\, dS = 1 - \int_{-\infty}^{\infty} \prod_{i=1}^{n} (1 - F_{R_i}(S_i)) f_S(S)\, dS$$

Therefore, the reliability index for the series system can be calculated by:

$$\beta = -\Phi^{-1}(P_f)$$

Consider a statically indeterminate structure; Failure in a single bar will not necessarily result in failure of the whole structure. A *failure mode* can be defined as follows: A set of failure members forms a *mechanism* which causes the structure to fail, then this set of failure members is called *failure mode*. Therefore, a failure mode can be represented by a parallel system. In practice, a redundent structure usually has a large number of different failure modes, and each failure mode will be modelled by a parallel system. Therefore, the failure modes (parallel systems) are joined in a series system as shown below:

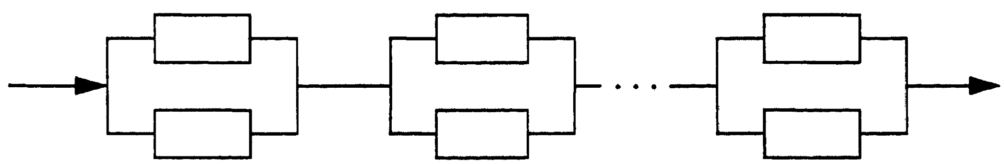


6.3MODELLING OF STRUCTURES

For some structures the reliability of the structure is calculated on the basis of failure of a single component, where the probability of failure of any component and the correlation between them are taken into account, and then all the failure compoonents are combined to make up the series system. The modelling of this series system is called *systems modelling at level 1* as shown:

Usually the probability of failure of structural systems can be estimated with sufficient accuracy only by considering a finite number of significant failure components.

For some structures the reliability of the structure is calculated on the basis of failure of a pair of components, where the probability of failure of any pair of components and the correlation between them are taken into account, and then all the failure pairs are combined to make up the series system. The modelling of this series system is called *system modelling at level 2*, where a failure mode is a parallel system with two failure components as shown:



failure mode 1 failure mode 2 failure mode n

In a same manner *system modelling at level N*, where *N=1,2,3,...* can be defined.

The most frequently used failure mode of structural systems is a mechanism which is modelled by a parallel system. These mechanisms are then combined in a series system; The modelling of this series system is called *system modelling at mechanism level.* Usually the number of mechanisms of structures is very large, and it is impossible to consider all possible mechanisms. Therefore, only some reasonable amount of significant mechanisms should be considered.

6.4CALCULATION OF THE MULTIVARIATE NORMAL DISTRIBUTION FUNCTION

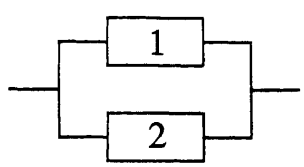
For a significant pair of failure components, estimation of the *bivariate normal distribution function* with zero mean values $\Phi_2(-\beta_1, -\beta_2;\rho)$, where ρ is the correlation coefficient between β_1 and β_2 is given by:

$$\Phi_2(-\beta_1, -\beta_2;\rho) = \int_{-\infty}^{\beta_1}\int_{-\infty}^{\beta_2}\varphi(x_1, x_2;\rho)\,dx_1 dx_2$$

where the bivariate normal density function with zero mean values is given by:

$$\varphi_2(x_1, x_2;\rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}(x_1^2 + x_2^2 - 2\rho x_1 x_2)}$$

This equation is useful in estimation of the probability of failure for a pair of failure components, where the corresponding safety margins M_1 and M_2 are linear.



$$M_1 = u_0 + u_1 Z_1 + \ldots + u_n Z_n$$

$$M_2 = v_0 + v_1 Z_1 + \ldots + v_n Z_n$$

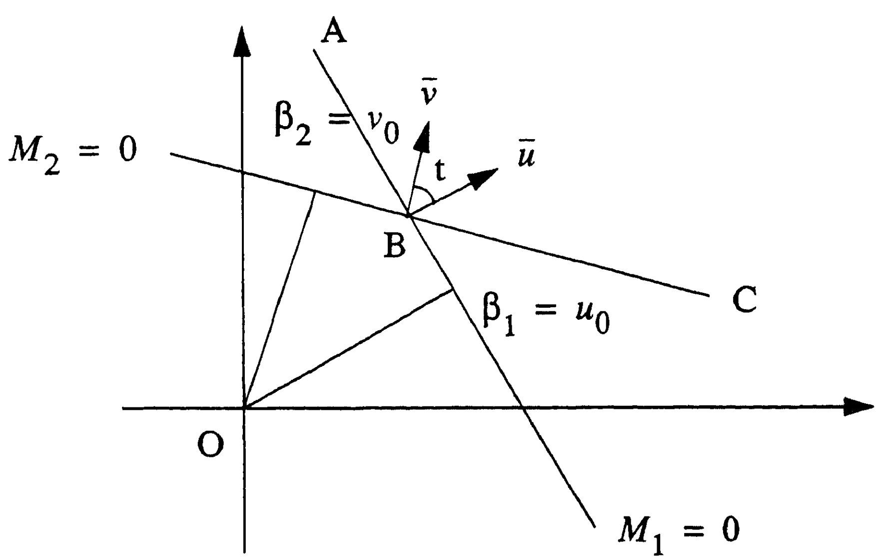
where Z_i, $i = 1, \ldots, n$ are standardized normally distributed and uncorrelated, and M_1 and M_2 are also standardized normally distributed with a correlation coefficient ρ:

$$\rho = \sum_{i=1}^{n} u_i v_i$$

$\bar{u} = (u_0, \ldots, u_n)$ and $\bar{v} = (v_0, \ldots, v_n)$ are unit normal vectors, and the correlation coefficient ρ can be written as

$$\rho = \cos t$$

where t is the angle between the unit vectors $\bar{u}$ and $\bar{v}$. The reliability indices β_1 and β_2 corresponding to the safety margins M_1 and M_2 are equal to u_0 and v_0.



The probability of failure P_f is equal to the intersection area of $M_1 \leq 0$ and $M_2 \leq 0$ in the angle ABC, and equal to

$$P_f = \Phi_2(-\beta_1, -\beta_2; \rho)$$

The n-dimensional multivariate standardized normal distribution function Φ_n is definded by

$$\Phi_n(-\bar{\beta};\bar{C}) = \int_{-\infty}^{-\beta_1} \cdots \int_{-\infty}^{-\beta_n} \frac{1}{(2\pi)^{n/2}|\bar{C}|^{1/2}} e^{-\frac{1}{2}\sum_{i,j=1}^{n}(x_i^T M_{ij} x_j)} dx_1 \ldots dx_n$$

where $-\bar{\beta} = (-\beta_1, \ldots, -\beta_n)$, $\bar{M} = \bar{C}^{-1}$, where $\bar{C}$ is the covariance matrix defined by:

$$\begin{bmatrix} Var[x_1] & Cov[x_1, x_2] & \ldots & Cov[x_1, x_n] \\ Cov[x_2, x_1] & Var[x_2] & \ldots & Cov[x_2, x_n] \\ \ldots & \ldots & \ldots & \ldots \\ Cov[x_n, x_1] & Cov[x_n, x_2] & \ldots & Var[x_n] \end{bmatrix}$$

In general the calculation of Φ_n for $n \geq 3$ can only be estimated in an approximate way.

Alternatively, Φ_n can be calculated by

$$\Phi_n(-\bar{\beta};\bar{\rho}) = \int_{-\infty}^{-\beta} \cdots \int_{-\infty}^{-\beta_n} \frac{1}{(2\pi)^{n/2}|\bar{\rho}|^{1/2}} e^{-\frac{1}{2}\sum_{i,j=1}^{n} x_i^T M_{ij} x_j} dx_1 \ldots dx_n$$

where $\bar{M} = \bar{\rho}^{-1}$, and the correlation matrix $\bar{\rho}$ is defined by:

$$\bar{\rho} = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix}$$

CHAPTER 7
RELIABILITY OF STRUCTURAL SYSTEMS

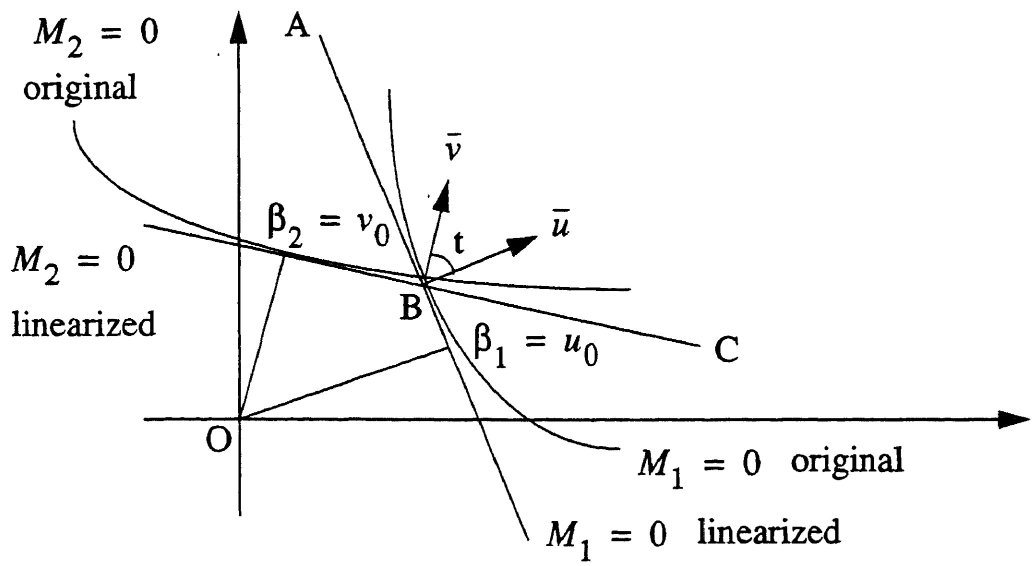
7.1PROBABILITY OF FAILURE OF SERIES SYSTEMS

It has been suggested that the reliability of a structure be estimated on the basis of a series system modelling, where the components are failure modes, and the failure modes are modelled by parallel systems. Now, consider a simple series system which consists of two failure modes denoted by the safety margins $M_1 = f_1(X_1, X_2)$ and $M_2 = f_2(X_1, X_2)$

If $F_i = \{M_i \leq 0\} = f_i(X_1, X_2) \leq 0, \quad i = 1, 2$

Then the probability of failure P_f of the series system is given by:

$$P_f = P(F_1 \cup F_2) = P(M_1 \leq 0 \cup M_2 \leq 0)$$



The safety margins M_1 and M_2 can be linearized in their respective reliability index points β_1 and β_2.

$$M_1 = \beta_1 + u_1 X_1 + u_2 X_2 = \beta_1 + u^T \bar{X}$$

$$M_2 = \beta_2 + v_1 X_1 + v_2 X_2 = \beta_2 + v^T \bar{X}$$

where $\bar{u} = (u_1, u_2)$, $\bar{v} = (v_1, v_2)$ are unit vectors.

Then the approximation of P_f can be given by:

$$P_f \approx P((M_1 \le 0) \cup (M_2 \le 0)) = P((\beta_1 + u^T \bar{X} \le 0) \cup (\beta_2 + v^T \bar{X} \le 0))$$

$$= P((u^T \bar{X} \le -\beta_1) \cup (v^T \bar{X} \le -\beta_2))$$

$$= 1 - P((u^T \bar{X} > -\beta_1) \cap (v^T \bar{X} > -\beta_2))$$

$$= 1 - P\left((-u^T \overline{X} < \beta_1) \cap (-v^T \overline{X} < \beta_2) \right) = 1 - \Phi_2 (\beta_1, \beta_2; \rho)$$

where X_1 and X_2 are independent standard normal variables, and ρ is the correlation coefficient given by:

$$\rho = \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2$$

Φ_2 is the bivariate normal distribution function.

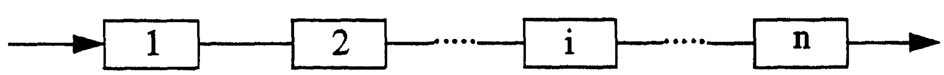
Therefore, the reliability index β for the whole series system can be given by:

$$\beta = -\Phi^{-1}(P_f) \approx -\Phi^{-1}\left(1 - \Phi_2 (\beta_1, \beta_2; \rho)\right)$$

Now, consider a general series system with n components as shown below and let the safety margin for component i be given by:

$$M_i = f_i(\overline{X}), i = 1, 2, \ldots, n$$

where $\overline{X} = (X_1, X_2, \ldots, X_m)$ are the basic random variables, and f_i are non-linear failure functions.



Usually, basic random variables $\overline{X} = (X_1, \ldots, X_m)$ are not independent; therefore, one should find out the corresponding uncorrelated variables $\overline{Y} = (Y_1, \ldots, Y_m)$ and then find out the corresponding independent standard normal variables

$\bar{Z} = (Z_1, \ldots, Z_m)$ (see chapter 5) so that the probability of failure P_{f_i} of component i can be evaluated as follows:

$$P_{f_i} = P(M_i \le 0) = P(f_i(\bar{X}) \le 0) = P(g_i(\bar{Z}) \le 0)$$

Then the approximation of P_{f_i} can be calculated by linearization of g_i at the reliability index β point.

$$P_{f_i} \approx P(g_i(\bar{Z}) \le 0) \approx P(\beta_i + \bar{u}_i^T \bar{Z} \le 0) = P(\bar{u}_i^T \bar{Z} \le -\beta_i) = \Phi(-\beta_i)$$

where $\bar{u}_i$ is the unit vector, β_i is reliability index, and Φ is the standard normal distribution function.

Therefore, the approximation of the probability of failure P_f of the series system can be estimated as follows:

$$P_f = P\left(\bigcup_{i=1}^{n} (M_i \le 0)\right) = P\left(\bigcup_{i=1}^{n} (f_i(\bar{X}) \le 0)\right) = P\left(\bigcup_{i=1}^{n} (g_i(\bar{Z}) \le 0)\right)$$

$$\approx P\left(\bigcup_{i=1}^{n} (\beta_i + \bar{u}_i^T \bar{Z} \le 0)\right) = P\left(\bigcup_{i=1}^{n} (\bar{u}_i^T \bar{Z} \le -\beta_i)\right)$$

$$= 1 - P\left(\bigcap_{i=1}^{n} (\bar{u}_i^T \bar{Z} > -\beta_i)\right) = 1 - P\left(\bigcap_{i=1}^{n} (-\bar{u}_i^T \bar{Z} < \beta_i)\right)$$

$$= 1 - \Phi_n(\bar{\beta}; \bar{\rho})$$

where $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ is the reliability indices, $\bar{\rho} = [\rho_{ij}]$ is the correlation matrix, $\rho_{ij} = \bar{u}_i^T \bar{u}_j$, Φ_n is the n-dimensional standardized normal distribution function.

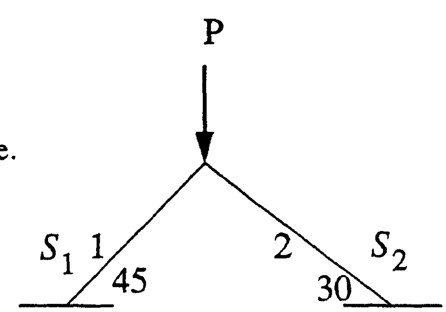
For a series system the estimate of the probability of failure, $P_f = 1 - \Phi_n(\bar{\beta};\bar{\rho})$, can be reduced to evaluate Φ_n, but for $n \geq 3$ the calculation can only be treated in the approximate method.

Example 1. A structure consists of two bars loaded by a concentrated load P as shown. Assume that the resistance strength in bar 1 is 1R, and in bar 2 is 2R. Assume that P and R are independent normally distributed random variables,with:

$\mu_P = 10KN \quad \mu_R = 10KN$

$\sigma_P = 2KN \quad \sigma_R = 1KN$

Calculate the probability of failure of the sturcture.



$$\sum F_y = 0 \quad S_1 \sin 45 + S_2 \sin 30 - P = 0$$

$$\sum F_x = 0 \quad S_1 \cos 45 - S_2 \cos 30 = 0$$

By solving these simultaneous equations, one can get:

$S_1 = 0.897P, \quad S_2 = 0.732P$

Therefore, the safety margins can be given by:

$$M_1 = R_1 - S_1 = R - 0.897P$$

$$M_2 = R_2 - S_2 = 2R - 0.732P$$

$$Z_1 = \frac{R - \mu_R}{\sigma_R} = \frac{R - 10}{1} \quad R = Z_1 + 10$$

$$Z_2 = \frac{P - \mu_P}{\sigma_P} = \frac{P - 10}{2} \quad P = 2Z_2 + 10$$

$$M_1 = R - 0.897P = (Z_1 + 10) - 0.897\,(2Z_2 + 10) = Z_1 - 1.794Z_2 + 1.03$$

$$M_2 = 2R - 0.732P = 2\,(Z_1 + 10) - 0.732\,(2Z_2 + 10)$$
$$= 2Z_1 - 1.464Z_2 + 12.68$$

$$M_1 = 0.501 + 0.487Z_1 - 0.873Z_2$$
$$M_2 = 5.12 + 0.807Z_1 - 0.591Z_2$$

where $\beta_1 = 0.5$ and $\beta_2 = 5.12$, and the correlation coefficient is

$$\rho = 0.487 \times 0.807 + 0.873 \times 0.591 = 0.909$$

Therefore, the probability of failure of the structure is

$$P_f = 1 - \Phi_2\,(\beta_1, \beta_2; \rho) = 1 - \Phi_2\,(0.5, 5.12, 0.91)$$

7.2APPROXIMATE TECHNIQUES FOR SERIES SYSTEMS

It is very difficult to calculate the value of the multinormal distribution function Φ_n, when n is greater than three; therefore, approximate techniques are needed, In this section two bounding methods are introduced.

First, *the simple bounds method* is suggested by *Thoft-Christensen* as follows:

$$\max_{i=1}^{n} \Phi(-\beta_i) \le P_f \le 1 - \prod_{i=1}^{n} (1 - \Phi(-\beta_i))$$

When safety margins are normally distributed and $\rho \ge 0$, the simple bounds can be used, but when the gap between lower and upper bounds is big this method is rarely used. The lower bound is the exact value of P_f when $\rho_{ij} = 1$ for all i and j are totally dependent; The upper bound is the exact value of P_f when $\rho_{ij} = 0$ for all $i \neq j$ are totally independent.

Example 2. Consider the structure of example 1, the probability of failure P_f of the structure is given as: $P_f = 1 - \Phi_2(\beta_1, \beta_2; \rho) = 1 - \Phi_2(0.5, 5.12; 0.91)$. Calculate the probability of failure P_f by using the simple bounds method.

$$P_{f_1} = \Phi(-\beta_1) = \Phi(-0.5) = 0.4801$$

$$P_{f_2} = \Phi(-\beta_2) = \Phi(-5.12) = 1.536\times10^{-7}$$

$$max\Phi(-\beta_i) = \Phi(-0.5) = 0.4801$$

$$1-\prod_{i=1}^{n}(1-\Phi(\beta_i)) = 1-(1-\Phi(-\beta_1))(1-\Phi(-\beta_2))$$

$$= 1-(1-0.4801)(1-1.536\times10^{-7}) = 0.480100079$$

Therefore, the bounds for the probability of failure P_f are:

$$0.4801 \le P_f \le 0.480100079$$

Second, the *Ditlevsen bounds* is defined as follows:

upper bound: $P_f \le \sum_{i=1}^{n}\Phi(-\beta_i) - \sum_{i=2,\ j<i}^{n} max\Phi_2(-\beta_i, -\beta_j, \rho)$

lower bound: $P_f \ge \Phi(-\beta_1) + \sum_{i=2}^{n} max\left[\Phi(-\beta_i) - \sum_{j=1}^{i-1}\Phi_2(-\beta_i, -\beta_j;\rho), 0\right]$

Note that ordering is important, where $P(-\beta_1) \ge P(-\beta_2) \ge \ldots \ge P(-\beta_n)$

The gap between the lower bound and upper bound of the *Ditlevsen bounds* is usually much smaller than the gap between the *simple bounds*.

Example 3. Consider the structure of example 1 again, the probability of failure P_f is given. Now use the Ditlevsen bounds to calculate P_f value.

$$P_f = 1 - \Phi_2(\beta_1, \beta_2; \rho) = 1 - \Phi_2(0.5, 5.12; 0.91)$$

$$P_{f_1} = \Phi(-\beta_1) = \Phi(-0.5) = 0.4801$$

$$P_{f_2} = \Phi(-\beta_2) = \Phi(-5.12) = 1.536\times10^{-7}$$

Upper bound: $P_f \le \Phi(-0.5) + \Phi(-5.12) - \Phi_2(-5.12, -0.5; 0.91)$

$$= 0.4801 + 1.536\times10^{-7} - 3.4029\times10^{-8} = 0.480100119$$

lower bound: $P_f \ge \Phi(-0.5) + max[\Phi(-5.12) - \Phi_2(-5.12, -0.5; 0.91), 0]$

$$= 0.4801 + 1.536\times10^{-7} - 3.4029\times10^{-8} = 0.480100119$$

$$0.480100119 \le P_f \le 0.480100119$$

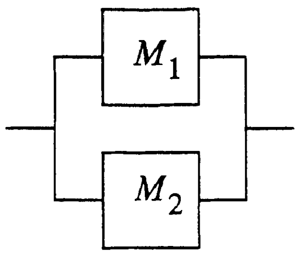
7.3PROBABILITY OF FAILURE OF PARALLEL SYSTEMS

It has been mentioned several time that the reliability of a structural system is modelled by a series system of parallel systems; Each parallel system represents a failure mode. Then next step is to calculate the probability of failure for each parallel system and the correlation between the parallel systems, and then final step is to calculate the probability of failure of the series system of parallel systems by the methods suggested in previous section.

Consider a simple parallel system(failure mode) with only two failure components and the safety margins are given by $M_1 = f_1(X_1, X_2)$ and $M_2 = f_2(X_1, X_2)$, where

X_1 and X_2 are independent standard normally distributed random variables. If failure functions $F_1 = (M_1 \leq 0)$, $F_2 = (M_2 \leq 0)$, then the probability of failure P_f of the parallel system can be given by:

$$P_f = P(F_1 \cap F_2)$$



$$M_1 = \beta_1 + u_1 X_1 + u_2 X_2 = \beta_1 + \bar{u}^T \bar{X}$$

$$M_2 = \beta_2 + v_1 X_1 + v_2 X_2 = \beta_2 + \bar{v}^T \bar{X}$$

$$P_f \approx P((M_1 \leq 0) \cap (M_2 \leq 0)) = P((\beta_1 + \bar{u}^T \bar{X} \leq 0) \cap (\beta_2 + \bar{v}^T \bar{X} \leq 0))$$

$$= P((\bar{u}^T \bar{X} \leq -\beta_1) \cap (\bar{v}^T \bar{X} \leq -\beta_2)) = \Phi_2(-\beta_1, -\beta_2; \rho)$$

where ρ the correlation coefficient is given by:

$$\rho = \sum_i u_i b_i = \bar{u}^T \bar{v} = u_1 v_1 + u_2 v_2$$

Φ_2 is the bivariate normal distribution function.

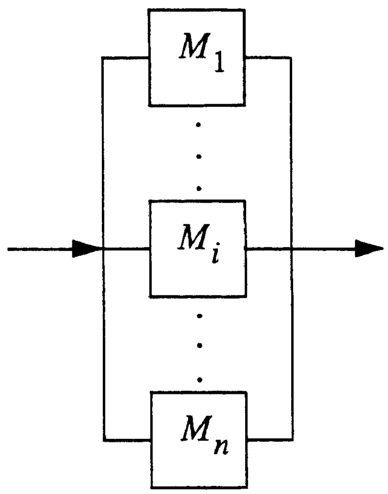
Therefore, the reliability index β for the parallel system can be obtained by:

$$\beta = -(\Phi^{-1}(P_f) \approx -\Phi^{-1}(-\beta_1, -\beta_2; \rho))$$

The formula derived above can be generalized to a general form where the parallel system has n failure components and where the number of basic varibales is m. Let the safety margin for component i be given by:

$$M_i = f_i(\overline{X}), i = 1, 2, \ldots, n$$

where $\overline{X} = (X_1, \ldots, X_m)$ are basic variables and where f_i are non-linear functions.



The probability of failure P_{f_i} of component i can be derived as before (see section 7.1) so that

$$P_{f_i} = P(M_i \leq 0) = P(f_i(\overline{X}) \leq 0) = P(g_i(\overline{Z}) \leq 0)$$

where the basic variables $\overline{X} = (X_1, \ldots, X_m)$ are transformed into independent standard normal variables $\overline{Z} = (Z_1, \ldots, Z_m)$.

The approximation of P_{f_i} can be estimated by linearizing g_i in the reliability index point β_i

$$P_{f_i} = P(g_i(\bar{Z}) \le 0) \approx P(\beta_i + \bar{u}_i^T \bar{Z} \le 0) = P(\bar{u}_i^T \bar{Z} \le -\beta_i) = \Phi(-\beta_i)$$

where $\bar{u}_i$ is the unit normal vector, β_i is the reliability index, and Φ is the standard normal distribution function.

Therefore, an approximation of the probability of failure P_f for the general parallel system can then be estimated as follows:

$$P_f = P\left(\bigcap_{i=1}^{n} (M_i \le 0)\right) = P\left(\bigcap_{i=1}^{n} (f_i(\bar{X}) \le 0)\right) = P\left(\bigcap_{i=1}^{n} (g_i(\bar{Z}) \le 0)\right)$$

$$\approx P\left(\bigcap_{i=1}^{n} (\beta_i + \bar{u}_i^T \bar{Z} \le 0)\right) = P\left(\bigcap_{i=1}^{n} (\bar{u}_i^T \bar{Z} \le -\beta_i)\right) = \Phi_n(-\bar{\beta};\bar{\rho})$$

where $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ and $\bar{\rho} = [\rho_{ij}]$ is the correlation matrix for the linearized safety margins, i.e. $\rho_{ij} = \sum_{i=j=1}^{m} \bar{u}_i \bar{u}_j = \bar{u}_i^T \bar{u}_j$. Φ_n is n-dimensional standardized normal distribution function.

From the formula $P_f = \Phi_n(-\bar{\beta};\bar{\rho})$, the estimation of the probability of failure of a parallel system with linear and normally distributed safety margins is reduced to estimate Φ_n. However, as mentioned before, estimation of Φ_n for n greater than three can only be treated in an approximate approach.

7.4APPROXIMATE TECHNIQUES FOR PARALLEL SYSTEMS

Since it is difficult to calculate the multinormal distribution function Φ_n directly, approximate methods must be considered.

The *simple bounds* is suggested by *Thoft-Christensen* as follows:

$$\prod_{i=1}^{n} \Phi(-\beta_i) \le P_f \le \min_{i=1}^{n} \Phi(-\beta_i)$$

The lower bound is the exact value of P_f when $\rho_{ij} = 0\ (i \ne j)$ are totally independent; The upper bound is the exact value of P_f when $\rho_{ij} = 1\ (for\ \text{all}\ i\ \text{and}\ j)$ are totally dependent.

The *modified simple bounds* is introduced by *Murotsu* as follows:

$$\prod_{i=1}^{n} \Phi(-\beta_i) \le P_f \le \min_{i,j=1}^{n} \Phi_2(-\beta_i, -\beta_j; \rho_{ij})$$

The *Hohenbichler approximation*:

$$\Phi_n(\bar{\beta};\bar{\rho}) \approx \Phi(\beta_1)\,\Phi_{n-1}(\bar{\beta}^{e}_{(2)};\bar{\rho}_{(2)})$$

where the equivalent reliability index $\bar{\beta}^{e}_{i}$ is defind by:

$$\beta_i^e = \beta_i^e(\bar{\varepsilon})\Big|_{\bar{\varepsilon}=0} = -\Phi^{-1}(P(f_i(\bar{Z}+\bar{\varepsilon}) \le 0))\Big|_{\bar{\varepsilon}=\bar{0}}, \text{where}$$

$\bar{Z} = (Z_1, \ldots, Z_n)$ are standard normal independent variables, and the corresponding $\bar{u}_i^e$ unit vector is definde by:

$$u_{ij}^e = \frac{\left.\frac{\partial}{\partial \varepsilon_j}\beta_i^e\right|_{\bar{\varepsilon}=\bar{0}}}{\sqrt{\sum_{l=1}^{n}\left(\left.\frac{\partial}{\partial \varepsilon_l}\beta_i^e\right|_{\bar{\varepsilon}=\bar{0}}\right)}} \qquad j = 1, 2, \ldots, n$$

where $\beta_{(2)i}^e = -\beta_i^e, i = 1, \ldots, n$ and $\rho_{(2)kl} = \bar{u}_k^{e^T}\bar{u}_l^e$, where

$$\bar{u}_k^{e^T} = (u_{k1}^e, \ldots, u_{kn}^e), k = 2, \ldots, n$$

Therefore,the calculation of Φ_n has been reduced to calculation of Φ_{n-1}. By repeating the same procedure, it gives the following approximation:

$$\Phi_n(\bar{\beta};\bar{\rho}) = \Phi(\beta_1)\Phi(\beta_{2(2)}^e)\ldots\Phi(\beta_{(n)n}^e)$$

When a parallel system (failure mode) consists of only two failure components the bounds for $\Phi_2(-\beta_1, -\beta_2;\rho)$ have been derived by *Thoft-Christensen* :

$$for \quad \rho > 0 \quad max(P_1, P_2) \le \Phi_2(-\beta_1, -\beta_2, \rho) \le P_1 + P_2$$

$$for \quad \rho \le 0 \quad 0 \le \Phi_2(-\beta_1 - \beta_2;\rho) \le min(P_1, P_2)$$

where $P_1 = \Phi(-\beta_2)\,\Phi\left(-\frac{\beta_1 - \rho\beta_2}{\sqrt{1-\rho^2}}\right)$

$$P_2 = \Phi(-\beta_1)\,\Phi\left(-\frac{\beta_2 - \rho\beta_1}{\sqrt{1-\rho^2}}\right)$$

7.5EQUIVALENT LINEAR SAFETY MARGIN FOR PARALLEL SYSTEMS

From the previous section the probability of failure P_f of the parallel system can be represented by: $P_f = \Phi_n(-\bar{\beta};\bar{\rho})$

where $\bar{\beta} = (\beta_1, \ldots, \beta_n)$ are the reliability indices of the failure components, and $\bar{\rho}$ is the correlation matrix.

When the reliability of a structural system is modelled by a series system of parallel systems, the reliability of the structure can be estimated by the following steps:

(1) calculate the probability of failure for each parallel system.

(2) calculate the correlationship between the parallel systems.

(3) calculate the probability of failure of the series system.

Consider a parallel system (failure mode) with n components and the safety margin for element i ,i=1,...,n is linear as follows:

$$M_i = \beta_i + u_{i1}Z_1 + \dots + u_{im}Z_m = \beta_i + \sum_{j=1}^{m} u_{ij}Z_j$$

where the basic variables Z_i, $i = 1, \dots, m$ are independent standard normal variables, $\bar{u}_i = (u_{i1}, \dots, u_{im})$ is a unit normal vector, and where β_i is the reliability index.

Therefore, the reliability index β for the parallel system can be obtained by:

$$\beta = -\Phi^{-1}(\Phi_n(-\bar{\beta};\bar{\rho}))$$

The reliability index β^e of the equivalent linear safety margin M^e is equal to the reliability index β of the parallel system so that the equivalent linear safety margin M^e has the same sensitivity as the parallel system when the basic variables change.

Let the basic variables $\bar{Z} = (Z_1, \dots, Z_m)$ increase by a small amount $\bar{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_m)$. The corresponding reliability index $\beta(\bar{\varepsilon})$ of the parallel system becomes:

$$\beta(\bar{\varepsilon}) = -\Phi\left(P\left(\bigcap_{i=1}^{n}\left(\sum_{j=1}^{m}\beta_i + u_{ij}(Z_j + \varepsilon_j) \le 0\right)\right)\right)$$

$$= -\Phi\left(P\left(\bigcap_{i=1}^{n}\left(\sum_{j=1}^{m}u_{ij}Z_j \le -\beta_i - u_{ij}\varepsilon_j\right)\right)\right)$$

$$= -\Phi^{-1}(\Phi_n(-\bar{\beta} - \bar{u}\bar{\varepsilon};\bar{\rho}))$$

The equivalent linear safety margin M^e is defined by:

$$M^e = \beta^e + u_1^e Z_1 + \ldots + u_m^e Z_m = \beta^e + \sum_{j=1}^{m} u_j^e Z_j$$

where $\beta^e = \beta$. By the same increase $\bar{\varepsilon}$ in the basic variables Z_i, $i = 1, \ldots, m$ the reliability index $\beta^e(\bar{\varepsilon})$ becomes:

$$\beta^e(\bar{\varepsilon}) = -\Phi^{-1}(\Phi(-\beta^e - \bar{u}^{e^T}\bar{\varepsilon})) = \beta^e + \bar{u}^{e^T}\bar{\varepsilon} = \beta^e + u_1^e\varepsilon_1 + \ldots + u_1^e\varepsilon_m$$

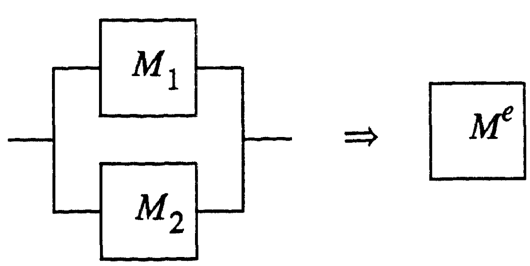
Let $\beta(\bar{0}) = \beta^e(\bar{0})$, one can get:

$$u_i^e = \frac{\left.\dfrac{\partial\beta}{\partial\varepsilon_i}\right|_{\bar{\varepsilon}=\bar{0}}}{\sqrt{\displaystyle\sum_{j=1}^{n}\left(\left.\dfrac{\partial\beta}{\partial\varepsilon_j}\right|_{\bar{\varepsilon}=\bar{0}}\right)^2}}, \quad i = 1, \ldots, m$$

Example 4. Assume that a parallel system (failure mode) consists of two failure components and the safety margins of the failure coimponents are given by:

$$M_1 = 3.0 - 0.3Z_1 + 0.954Z_2$$

$$M_2 = 3.5 - 0.866Z_1 + 0.5Z_2$$



where Z_1 and Z_2 are independent standard normal variables.

The reliability index β of the parallel system is :

$$\Phi(-\beta) = \Phi_2(-3.0, -3.5; 0.74) = 0.00008491; \beta = 3.76$$

$$\beta = -\Phi^{-1}(-3.0, 3.5; 0.74) = 3.76$$

let $\varepsilon_i = 0.1, i = 1, 2.$ for $\bar{\varepsilon} = (0.1, 0)$ one gets:

$$-\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -3.0 \\ -3.5 \end{bmatrix} - \begin{bmatrix} -0.3 & 0.954 \\ -0.866 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2.97 \\ -3.14 \end{bmatrix}$$

$$\beta(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.97, -3.14; 0.74)) = 3.53$$

for $\bar{\varepsilon} = (0, 0.1)$

$$-\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -3.0 \\ -3.5 \end{bmatrix} - \begin{bmatrix} -0.3 & 0.954 \\ 0.866 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -3.1 \\ -3.55 \end{bmatrix}$$

$$\beta(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-3.1, -3.55; 0.74)) = 3.98$$

Therefore,

$$\left.\frac{\partial \beta}{\partial \varepsilon_1}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{\beta(\bar{\varepsilon}) - \beta}{\varepsilon_1} = \frac{3.53 - 3.76}{0.1} = -2.3$$

$$\left.\frac{\partial\beta}{\partial\varepsilon_2}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{3.98 - 3.76}{0.1} = 2.2$$

By normalizing

:

$$\bar{u}^e = (u_1^e, u_2^e) = \left(\frac{-2.3}{\sqrt{(-2.3)^2 + 2.2^2}}, \frac{2.2}{\sqrt{(-2.3)^2 + 2.2^2}}\right) = (-0.7226, 0.6912)$$

Then the equivalent safety margin is given by:

$$M^e = \beta^e + u_1^e Z_1 + u_2^e Z_2 = 3.76 - 0.7226 Z_1 + 0.6912 Z_2$$

CHAPTER 8
GENERATION OF SAFETY MARGINS BY ANSYS

8.1INTRODUCTION

A real structure usually has many different modes of failure, In order to estimate the structural reliability, these failure modes and their corresponding safety margins must be given. For a simple structure the safety margins can be built up by hand calculation. In the conventional analysis, the structural safety margins are built up by using the principle of virtual work, but in practice, for a complex structure with large redundancy it is difficult to derive the safety margins by using the principle of virtual work. However, in this chapter, we show how to use ANSYS (a finite element software produced by Swanson Analysis Systems, Inc.) to derive the failure modes and their corresponding safety margins in detail.

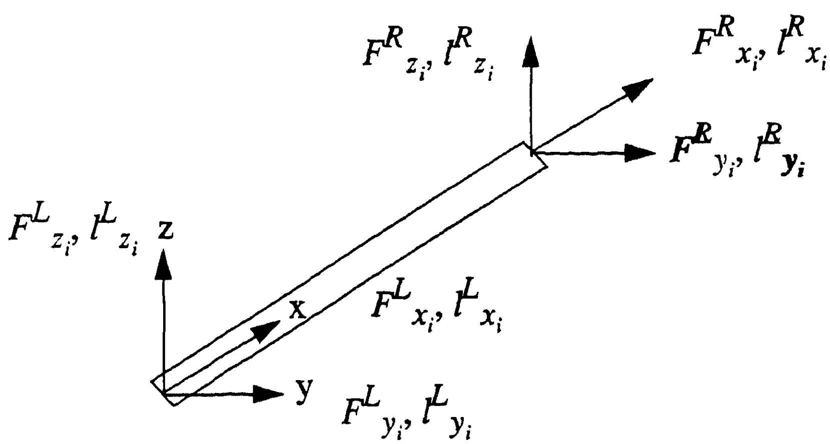
8.2THE THEORY OF GENERATION OF SAFETY MARGINS

Consider a structure with n bars. A bar will fail if the internal force exceeds the strength of the bar. The safety margin is determined by the difference between the strength of material and the internal force: $M_i = R_i - S_i$

where M_i is the safety margin of the i^{th} bar, R_i is the strength capacity of the i^{th} bar, and S_i is the internal force of the i^{th} bar.

The strength capacity R_i is given by specifying the material, and the internal force S_i can be evaluated by ANSYS finite element software.

Let $\bar{f}_i$ and $\bar{\delta}_i$ represent the nodal force vector and displacement vector of the i^{th} bar in the local coordinate system as shown below:



The stiffness equation of bar i is given by:

$$\bar{f}_i = \bar{k}_i \bar{\delta}_i$$

where $\bar{f}_i = (F^R_{x_i}, F^R_{y_i}, F^R_{z_i}, F^L_{x_i}, F^L_{y_i}, F^L_{z_i})$, $\bar{\delta}_i = (l^R_{x_i}, l^R_{y_i}, l^R_{z_i}, l^L_{x_i}, l^L_{y_i}, l^L_{z_i})$

Since $\sigma = E\varepsilon, \frac{F}{A} = E\frac{\Delta l}{l}, F = \frac{AE}{l}\Delta l = \frac{AE}{l}\delta = k\delta,$

nodal forces: $F^L_{x_i} = \frac{A_i E_i}{l_i}(\delta^L{}_{x_i} - \delta^R{}_{y_i})$,$F^R_{x_i} = \frac{A_i E_i}{l_i}(\delta^R{}_{x_i} - \delta^L{}_{y_i})$

$\bar{f}_i = \bar{k}_i \bar{\delta}_i$ can be written as the matrix form:

$$\begin{bmatrix} F^L_{x_i} \\ F^L_{y_i} \\ F^L_{z_i} \\ F^R_{x_i} \\ F^R_{y_i} \\ F^R_{z_i} \end{bmatrix} = \frac{A_i E_i}{l_i} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l^L_{x_i} \\ l^L_{y_i} \\ l^L_{z_i} \\ l^R_{x_i} \\ l^R_{y_i} \\ l^R_{z_i} \end{bmatrix}$$

where A_i is the cross area, E_i is the Young's modulus, l_i is the length of bar i.

Therefore, the stiffness matrix of bar i is given by:

$$\bar{k}_i = \frac{A_i E_i}{l_i} \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The displacement and nodal force vectors can be transformed into the global coordinate system by the transformation matrix $\bar{T}_i$.

$$\bar{\delta}_i = \bar{T}_i\bar{d}_i;\bar{d}_i = \bar{T}_i^{-1}\bar{\delta}_i$$

$$\bar{f}_i = \bar{T}_i\bar{F}_i;\bar{F}_i = \bar{T}_i^{-1}\bar{f}_i$$

where $\bar{d}_i$ and $\bar{F}_i$ are the displacement and nodal force vectors of bar i in the global coordinate system.

Therefore, the stiffness equation of bar i in the global coordinate system can be written as:

$$\bar{F}_i = \bar{K}_i\bar{d}_i$$

$$\bar{F}_i = \bar{T}_i^{-1}\bar{f}_i = \bar{T}_i^{-1}\bar{k}_i\bar{\delta}_i = \bar{T}_i^{-1}\bar{k}_i\bar{T}_i\bar{d}_i = \bar{K}_i\bar{d}_i$$

where $\bar{K}_i = \bar{T}_i^{-1}\bar{k}_i\bar{T}_i = \bar{T}_i^T\bar{k}_i\bar{T}_i$

In the similar manner the stiffness equations of other bars can be formed, and then the global nodal displacement vector $\bar{d}$ is formed by arranging the individual displacement vectors $\bar{d}_i$, and the global nodal force vector $\bar{F}$ is also formed by the individual nodal vectors $\bar{f}_i$, Furthermore, the whole structural stiffness matrix $\bar{K}$ is constructed by superposing the individual stiffness matrices.

Therefore, the total structure stiffness equation can be given by

$$\bar{F} = \bar{K}\bar{d}$$

The nodal force $\bar{f}_i$ in the local coordinate system is related to $\bar{F}_i$ in the global coordinate system; therefore, the nodal force $\bar{f}_i$ can be solved as follows:

$$\bar{f}_i = \bar{T}_i\bar{F}_i = \bar{T}_i\bar{K}_i\bar{d}_i = \bar{T}_i(\bar{T}_i^{-1}\bar{k}_i\bar{T}_i)\bar{K}_i^{-1}\bar{F} = (\bar{k}_i\bar{T}_i\bar{K}_i^{-1})\bar{F} = \bar{A}_i\bar{F}$$

where $\bar{A}_i = \bar{k}_i\bar{T}_i\bar{K}_i^{-1}$, and $\bar{K}_i^{-1}$ is a submatrix of $\bar{K}^{-1}$ corresponding to the i^{th} bar.

For a truss structure the local internal force is equal to the local axial force $S_i = F^R_{x_i} = -F^L_{x_i}$, and it can be written as: $S_i = \sum_{j=1}^{n} a_{ij}F_j$

where a_{ij} is the element of matrix $\bar{A}_i$ referred to S_i and F_j.

Therefore, the safety margin of the i^{th} bar can be defined by

$$M_i = R_i - S_i = R_i - \sum_{j=1}^{n} a_{ij}F_j$$

When the bar fails in tension or compression, the yield stress is taken into account, and when the bar is instable in compression, the bucking stress is considered.

For a statically determinate structure, the structure will fail, if any bar of the structure fails. For a statically indeterminate structure, the structure will not necessarily fail if any bar of the structure fails, and the failure will occure only after the structure becomes a mechanism. Failure modes will be produced by the following method. When any one bar fails, the internal forces will be redistributed among the survival bars and a bar next to fail is found. After any bar failed, the residual strength R_i is applied as an artificial force at the corresponding nodes, and its individual local stiffness matrix is set to zero. Repeating the same procedure, structural failure occurs when the failed bars reach some specified number q. A mechanism will be formed if the determinant of the total structure stiffness matrix

$\overline{K}^{(n-q)}$ becomes singular. That is

$$\left|\overline{K}^{(n-q)}\right| = 0$$

where *n-q* is the number of surviving bars.

The safety margins of the surviving bars after some bars failed can be defined as follows:

$$S_i^{n-q} = \sum_{j=1}^{n} a_{ij}^{(n-q)} F_j^{(n-q)} = \sum_{j=1}^{n} a_{ij}^{(n-q)} F_j - b_{ir_1}^{(n-q)} R_{r_1} -$$

$$b_{ir_2}^{(n-q)} R_{r_2} - \ldots - b_{ir_q}^{(n-q)} R_{r_q}$$

where $b_{ij}^{(n-q)}$ are the coefficients of influence and where suffix $(r_1, r_2, \ldots, r_q)$ denote the failed bars. Therefore, the safety margins are defined by:

$$M_i^{n-q} \equiv R_i - S_i^{n-q}, \text{ where } i = (1, 2, \ldots, n-q-1).$$

Therefore, a structural failure criterion for a statically indeterminate(redundant) truss can be defined as:

$$M_i^{n-q} \leq 0$$

8.3GENERATION OF SAFETY MARGINS USING ANSYS

Usually each bar can fail in two different forms, namely in *tension* or in *compression*. Let R_i^+ and R_i^- represent the strength capacity in tension and in compression for bar i and let S_i represent the load effect of bar i. Then the following two safety margins are described:

$M_i^+ = R_i^+ - S_i$ tension load effect

$M_i^- = R_i^- + S_i$ compression load effect

Therefore, the corresponding safety margin M_i is determined by:

$$M_i = min\,(R_i^+ - S_i, R_i^- + S_i)$$

Example 1. Consider a statically determinate truss structure as shown below. Assume that the strength capacities for compression and tension are same $R_i^+ = R_i^-$. Calculate the safety margins of the total structure.

The ANSYS PROGRAMS and the *corresponding results* are listed in the appendixes A, A1, and A2.

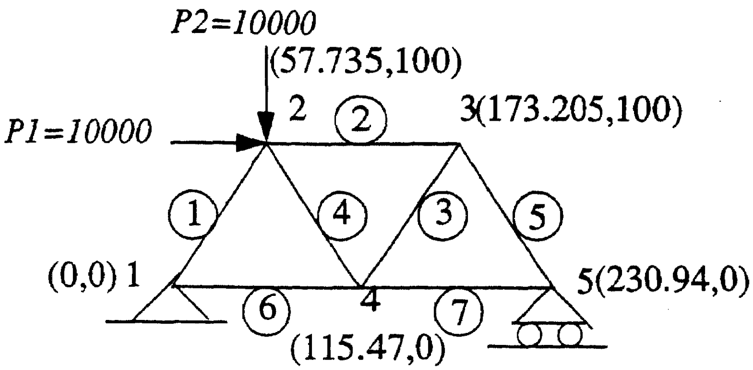


figure a

$b + r = 2j - e \Rightarrow 7 + 3 = 2 \times 5 - 0$ (statically determinate)

The external loads in figure a are the linear combinations of individual external loads as shown in figure a1 and in figure a2.

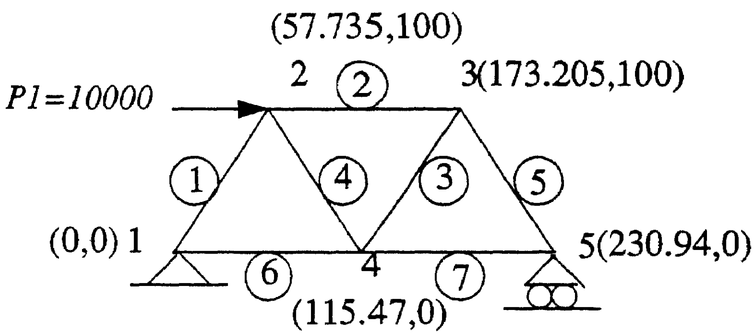


figure a1

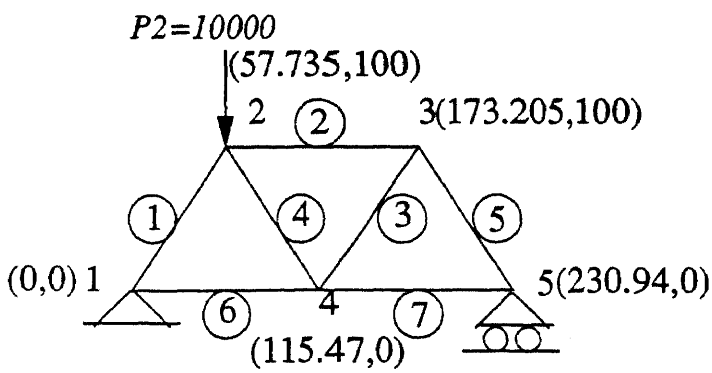


figure a2

The load effects of figure a1 are as follows:

	ELEMENT	FORCE
1	5000	tension
2	-5000	compression
3	5000	tension
4	-5000	compression
5	-5000	compression
6	7500	tension
7	2500	tension

Therefore, for each element the safety margins of figure a1 are given by:

$$M_1 = R_1^{+} - S_1\,(tension) = R_1 - 5000 = R_1 - 0.5P_1$$

$$M_2 = R_2^{-} - S_1\,(compression) = R_2 - 5000 = R_2 - 0.5P_1$$

$$M_3 = R_3^{+} - S_3\,(tension) = R_3 - 5000 = R_3 - 0.5P_1$$

$$M_4 = R_4^{-} + S_4\,(compression) = R_4 - 5000 = R_4 - 0.5P_1$$

$$M_5 = R_5^{-} + S_5\,(compression) = R_5 - 5000 = R_5 - 0.5P_1$$

$$M_6 = R_6^{+} - S_6\,(tension) = R_6 - 7500 = R_6 - 0.75P_1$$

$$M_7 = R_7^{+} - S_7\,(tension) = R_7 - 2500 = R_7 - 0.25P_1$$

The load effects of figure a2 are as follows:

	ELEMENT	FORCE
1	-8660.3	compression
2	-2886.8	compression
3	2886.8	tension
4	-2886.8	compression
5	-2886.8	compression
6	4330.1	tension
7	1443.4	tension

Therefore, for each element the safety margins of figure a2 are given by:

$$M_1 = R_1^- + S_1\,(compression) = R_1 - 8660.3 = R_1 - 0.866P_2$$

$$M_2 = R_2^- + S_2\,(compression) = R_2 - 2886.8 = R_2 - 0.289P_2$$

$$M_3 = R_3^{\,+} - S_3\,(tension) = R_3 - 2886.8 = R_3 - 0.289P_2$$

$$M_4 = R_4^- + S_4\,(compression) = R_4 - 2886.8 = R_4 - 0.289P_2$$

$$M_5 = R_5^- + S_5\,(compression) = R_5 - 2886.8 = R_5 - 0.289P_2$$

$$M_6 = R_6^{\,+} - S_6\,(tension) = R_6 - 4330.1 = R_6 - 0.433P_2$$

$$M_7 = R_7^{\,+} - S_7\,(tension) = R_7 - 1443.4 = R_7 - 0.144P_2$$

The load effects of figure a are as follows:

ELEMENT	FORCE

1	-3660.3	compression
2	-7886.8	compression
3	7886.8	tension
4	-7886.8	compression
5	-7886.8	compression
6	11830	tension
7	3943.4	tension

Therefore, the safety margins of the total sructure are given by the linear combinations of figure a1 and a2:

$$M_1 = R_1^- + S_1\,(compression) = R_1 - 3660.3 = R_1 + 0.5P_1 - 0.866P_2$$

$$M_2 = R_2^- + S_2\,(compression) = R_2 - 7886.8 = R_2 - 0.5P_1 - 0.289P_2$$

$$M_3 = R_3{}^+ - S_3\,(tension) = R_3 - 7886.8 = R_3 - 0.5P_1 - 0.289P_2$$

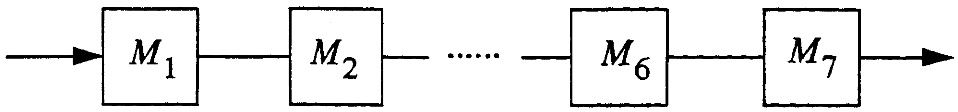
$$M_4 = R_4^- + S_4\,(compression) = R_4 - 7886.8 = R_4 - 0.5P_1 - 0.289P_2$$

$$M_5 = R_5^- + S_5\,(compression) = R_5 - 7886.8 = R_5 - 0.5P_1 - 0.289P_2$$

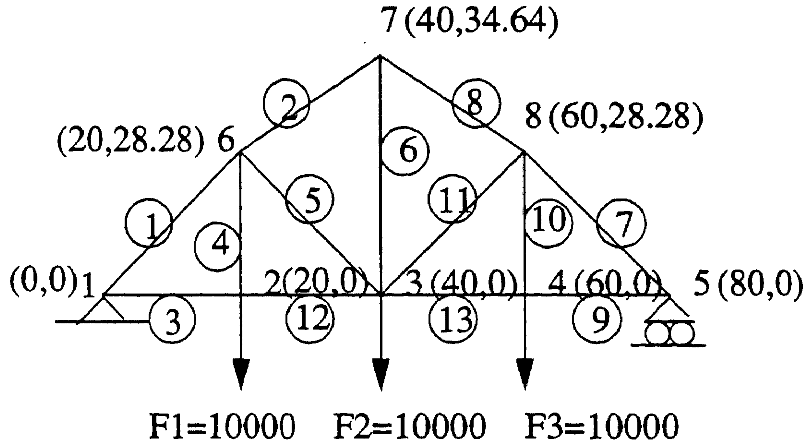
$$M_6 = R_6{}^+ - S_6\,(tension) = R_6 - 11830 = R_6 - 0.75P_1 - 0.433P_2$$

$$M_7 = R_7{}^+ - S_7\,(tension) = R_7 - 3943.4 = R_7 - 0.25P_1 - 0.144P_2$$

For a statically determinate structure, the failure modes can be plotted in a series system.



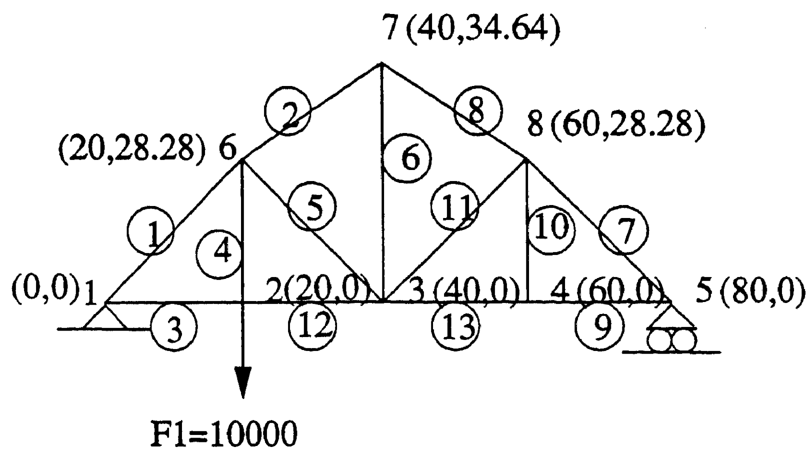
Example 2. Consider a statically determinate truss structure as shown below. Assume that the strength capacities for compression and tension are same and then calculate the safety margins of the total structure.



(Figure b)

The ANSYS PROGRAM and the corresponding results are listed in the appendixes B,B1,B2, and B3.

The external loads in figure b are the linear combinations of individual external loads as shown in figure b1,b2,and b3.



(Figure b1)

The load effects of figure b1 are as follows:

	ELEMENT	FORCE
1	-9185.6	compression
2	-3029.1	compression
3	5303.3	tension
4	10000	tension
5	-4185.6	compression
6	1835	tension
7	-3061.9	compression
8	-3029.1	compression
9	1767.8	tension
10	0	
11	1938.1	tension
12	5303.3	tension
13	1767.8	tension

Therefore, for each element the safety margins of figure b1 are given by:

$$M_1 = R_1^- + S_1\,(compression) = R_1 - 9185.6 = R_1 - 0.9186F_1$$

$$M_2 = R_2^- + S_2\,(compression) = R_2 - 3029.1 = R_2 - 0.3029F_1$$

$$M_3 = R_3^+ - S_3\,(tension) = R_3 - 5303.3 = R_3 - 0.5303F_1$$

$$M_4 = R_4^+ - S_4\,(tension) = R_4 - 10000 = R_4 - F_1$$

$$M_5 = R_5^- + S_5\,(compression) = R_5 - 4185.6 = R_5 - 0.4186F_1$$

$$M_6 = R_6^+ - S_6\,(tension) = R_6 - 1835 = R_6 - 0.1835F_1$$

$$M_7 = R_7^- + S_7\,(compression) = R_7 - 3061.9 = R_7 - 0.3062F_1$$

$$M_8 = R_8^- + S_8\,(compression) = R_8 - 3029.1 = R_8 - 0.3029F_1$$

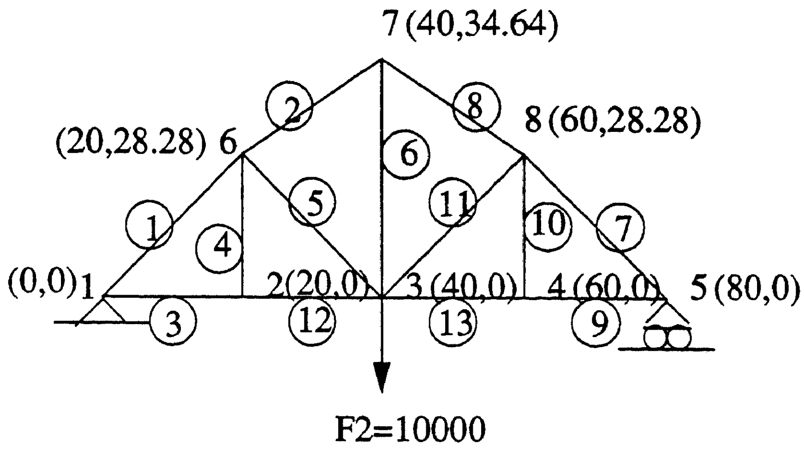
$$M_9 = R_9^+ - S_9\,(tension) = R_9 - 1767.8 = R_9 - 0.1768F_1$$

$$M_{10} = R_{10} - S_{10} = R_{10} - 0 = R_{10}$$

$$M_{11} = R_{11}^+ - S_{11}\,(tension) = R_{11} - 1938.1 = R_{11} - 0.1938F_1$$

$$M_{12} = R_{12}^+ - S_{12}\,(tension) = R_{12} - 5303.3 = R_{12} - 0.5303F_1$$

$$M_{13} = R_{13}^+ - S_{13}\,(tension) = R_{13} - 1767.8 = R_{13} - 0.1768F_1$$



(Figure b2)

The load effects of figure b2 are as follows:

	ELEMENT	FORCE
1	-6123.7	compression
2	-6058.1	compression
3	3535.5	tension
4	0	
5	3876.3	tension
6	3670	tension
7	-6123.7	compression
8	-6058.1	compression
9	3535.5	tension
10	0	

11	3876.3	tension
12	3535.3	tesnion
13	3535.3	tension

Therefore, for each element the safety margins of figure b2 are given by:

$$M_1 = R_1^- + S_1\,(compression) = R_1 - 6123.7 = R_1 - 0.6124F_2$$

$$M_2 = R_2^- + S_2\,(compression) = R_2 - 6058.1 = R_2 - 0.6058F_2$$

$$M_3 = R_3{}^+ - S_3\,(tesnion) = R_3 - 3535.5 = R_3 - 0.3536F_2$$

$$M_4 = R_4 - S_4 = R_4 - 0 = R_4$$

$$M_5 = R_5{}^+ - S_5\,(tesnion) = R_5 - 3876.3 = R_5 - 0.3876F_2$$

$$M_6 = R_6{}^+ - S_6\,(tesnion) = R_6 - 3670 = R_6 - 0.3670F_2$$

$$M_7 = R_7^- + S_7\,(compression) = R_7 - 6123.7 = R_7 - 0.6124F_2$$

$$M_8 = R_8^- + S_8\,(compression) = R_8 - 6058.1 = R_8 - 0.6058F_2$$

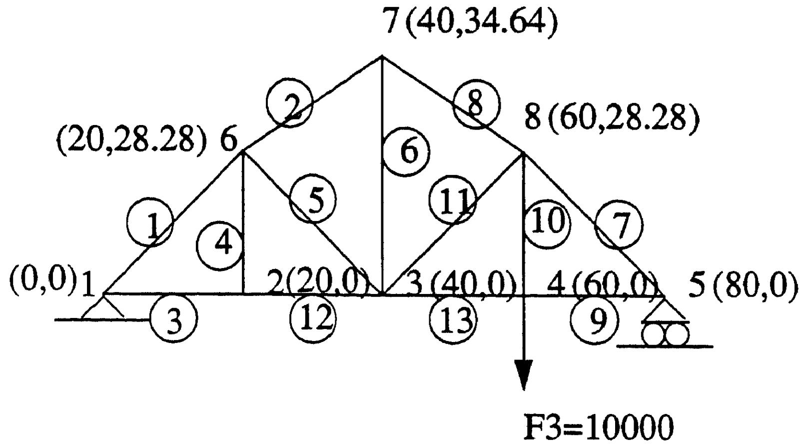
$$M_9 = R_9{}^+ - S_9\,(tesnion) = R_9 - 3535.5 = R_9 - 0.3536F_2$$

$$M_{10} = R_{10} - S_{10} = R_{10} - 0 = R_{10}$$

$$M_{11} = R_{11}^+ - S_{11}\,(tesnion) = R_{11} - 3876.3 = R_{11} - 0.3876F_2$$

$$M_{12} = R_{12}^+ - S_{12}\,(tesnion) = R_{12} - 3535.5 = R_{12} - 0.3536F_2$$

$$M_{13} = R_{13}^+ - S_{13}\,(tesnion) = R_{13} - 3535.5 = R_{13} - 0.3536F_2$$



(Figure b3)

The load effects of figure b3 are as follows:

	ELEMENT	FORCE
1	-3061.9	compression
2	-3029.1	compression
3	1767.8	tension
4	0	
5	1938.1	tension
6	1835	tension
7	-9185.6	compression
8	-3029.1	compression
9	5303.3	tension
10	10000	tension

11	-4185.6	compression
12	1767.8	tension
13	5303.3	tension

Therefore, for each element the safety margins of figure b3 are given by:

$$M_1 = R_1^- + S_1\,(compression) = R_1 - 3061.9 = R_1 - 0.3062F_3$$

$$M_2 = R_2^- + S_2\,(compression) = R_2 - 3029.1 = R_2 - 0.3029F_3$$

$$M_3 = R_3{}^+ - S_3\,(tension) = R_3 - 1767.8 = R_3 - 0.1768F_3$$

$$M_4 = R_4 - S_4 = R_4 - 0 = R_4$$

$$M_5 = R_5{}^+ - S_5\,(tension) = R_5 - 1938.1 = R_5 - 0.1938F_3$$

$$M_6 = R_6{}^+ - S_6\,(tension) = R_6 - 1835 = R_6 - 0.1835F_3$$

$$M_7 = R_7^- + S_7\,(compression) = R_7 - 9185.6 = R_7 - 0.9186F_3$$

$$M_8 = R_8^- + S_8\,(compression) = R_8 - 3029.1 = R_8 - 0.3029F_3$$

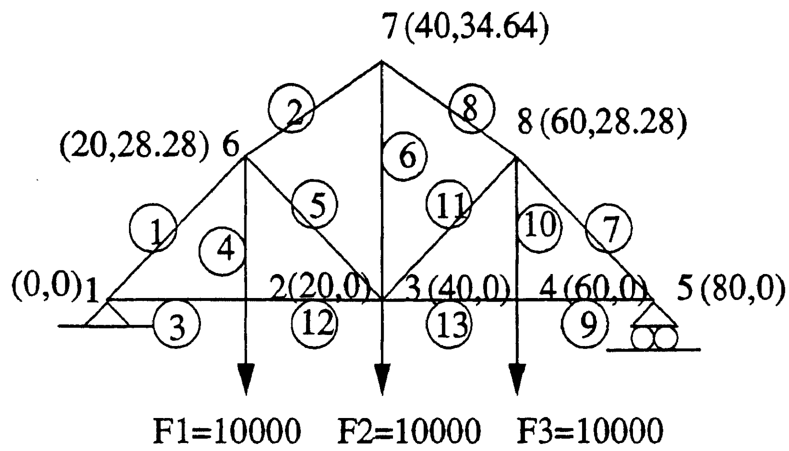
$$M_9 = R_9{}^+ - S_9\,(tension) = R_9 - 5303.3 = R_9 - 0.5303F_3$$

$$M_{10} = R_{10}^+ - S_{10}\,(tension) = R_{10} - 10000 = R_{10} - F_3$$

$$M_{11} = R_{11}^- + S_{11}\,(compression) = R_{11} - 4185.6 = R_{11} - 0.4186F_3$$

$$M_{12} = R_{12}^+ - S_{12}\,(tension) = R_{12} - 1767.8 = R_{12} - 0.1768F_3$$

$$M_{13} = R_{13}^{+} - S_{13}\,(tension) = R_{13} - 5303.3 = R_{13} - 0.5303F_3$$



(Figure b)

The load effects of figure b are as follows:

	ELEMENT	FORCE
1	-18371	compression
2	-12116	compression
3	10607	tension
4	10000	tension
5	1628.9	tension
6	7340.1	tension
7	-18371	compression
8	-12116	compression
9	10607	tension

10	10000	tension
11	1628.9	tension
12	10607	tension
13	10607	tension

Therefore, the safety margins of the total structure are given by the linear combinations of figure b1,b2,and b3:

$$M_1 = R_1^- + S_1(c) = R_1 - 18371 = R_1 - 0.9186F_1 - 0.6124F_2 - 0.3062F_3$$

$$M_2 = R_2^- + S_2(c) = R_2 - 12116 = R_2 - 0.3029F_1 - 0.6058F_2 - 0.3029F_3$$

$$M_3 = R_3{}^+ - S_3(t) = R_3 - 10607 = R_3 - 0.53F_1 - 0.3536F_2 - 0.1768F_3$$

$$M_4 = R_4{}^+ - S_4(t) = R_4 - 10000 = R_4 - F_1$$

$$M_5 = R_5{}^+ - S_5(t) = R_5 - 1628.9 = R_5 + 0.42F_1 - 0.3876F_2 - 0.1938F_3$$

$$M_6 = R_6{}^+ - S_6(t) = R_6 - 7340.1 = R_6 - 0.184F_1 - 0.367F_2 - 0.184F_3$$

$$M_7 = R_7^- + S_7(c) = R_7 - 18371 = R_7 - 0.3062F_1 - 0.6124F_2 - 0.9186F_3$$

$$M_8 = R_8^- + S_8(c) = R_8 - 12116 = R_8 - 0.3029F_1 - 0.6058F_2 - 0.3029F_3$$

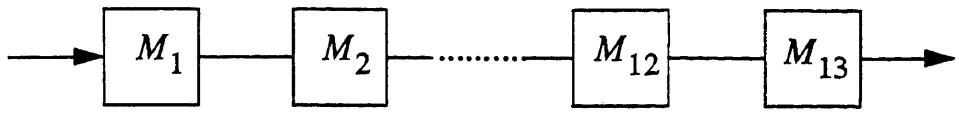
$$M_9 = R_9{}^+ - S_9(t) = R_9 - 10607 = R_9 - 0.1768F_1 - 0.3536F_2 - 0.53F_3$$

$$M_{10} = R_{10}^+ - S_{10}(t) = R_{10} - 10000 = R_{10} - F_3$$

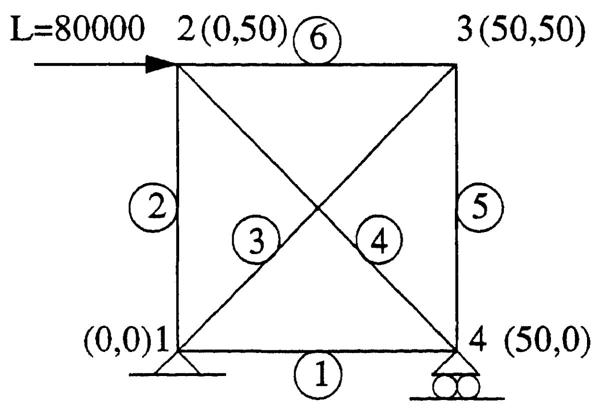
$$M_{11} = R_{11}^+ - S_{11}(t) = R_{11} - 1628.9 = R_{11} - 0.194F_1 - 0.39F_2 + 0.42F_3$$

$$M_{13} = R_{13}^+ - S_{13}(t) = R_{13} - 10607 = R_{13} - 0.53F_1 - 0.35F_2 - 0.18F_3$$

For a statically determinate structure, the failure modes can be plotted in a series system:



Example 3. Consider a redundant truss sturcture as shown below. Assume that the strength capacities for compression and tension are same $R_i^+ = R_i^- = 60000$ and then calculate the safety margins of the total structure.



$b + r > 2j - e \qquad 6 + 3 > 2 \times 4 - 0 \qquad 9 > 8$ (one redundancy).

The ANSYS PROGRAM and the corresponding results are listed in the appendixes C,C1,C2,C3,C4,C5,C6,D,D1,D2,D3,D4,D5,E,E1,E2,E3,E4,and E5. The force distributions are listed below:

			ELEMENT AND FORCE					
CASE	LOAD	APP.	1	2	3	4	5	6
1	L	C	40000	40000	56569	-56569	-40000	-40000
2	L	D	FAIL	0	113140	0	-80000	-80000
3	R1	E	FAIL	60000	-84853	-84853	60000	60000
4	L,R1	C1	FAIL	60000	28284	-84853	-20000	-20000
5	L	D1	0	FAIL	113140	0	-80000	-80000
6	R2	E1	60000	FAIL	-84853	-84853	60000	60000
7	L,R2	C2	60000	FAIL	28284	-84853	-20000	-20000
8	L	D2	80000	80000	FAIL	-113140	0	0
9	R3	E2	-42426	-42426	FAIL	60000	-42426	-42426
10	L,R3	C3	37574	37574	FAIL	-53137	-42426	-42426
11	L	D3	0	0	113140	FAIL	-80000	-80000
12	R4	E3	42426	42426	-60000	FAIL	42426	42426
13	L,R4	C4	42426	42426	53137	FAIL	-37574	-37574
14	L	D4	80000	80000	0	-113140	FAIL	0
15	R5	E4	-60000	-60000	84853	84853	FAIL	-60000
16	L,R5	C5	20000	20000	84853	-28284	FAIL	-60000
17	L	D5	80000	80000	0	-113140	0	FAIL
18	R6	E5	-60000	-60000	84853	84853	-60000	FAIL
19	L,R6	C6	20000	20000	84853	-28284	-60000	FAIL

From the above table and appendix C, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:

$$M_1 = R_1^{\ +} - S_1\,(tension) = R_1 - 40000 = R_1 - 0.5L$$

$$M_2 = R_2^{\ +} - S_1\,(tension) = R_2 - 40000 = R_2 - 0.5L$$

$$M_3 = R_3^{\ +} + S_3\,(tension) = R_3 - 56569 = R_3 - 0.7071L$$

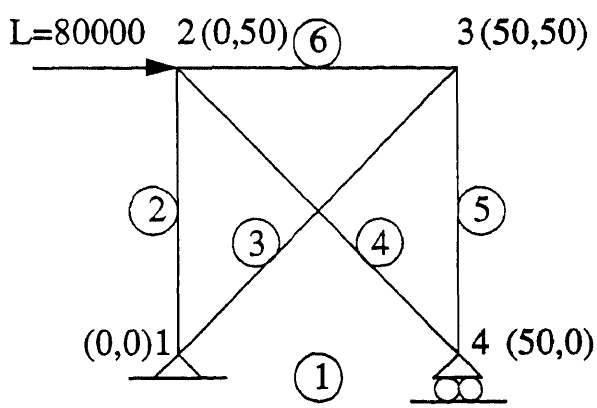
$$M_4 = R_4^{-} + S_4\,(compression) = R_4 - 56569 = R_4 - 0.7071L$$

$$M_5 = R_5^{-} + S_5\,(compression) = R_5 - 40000 = R_5 - 0.5L$$

$$M_6 = R_6^{-} + S_6\,(compression) = R_6 - 40000 = R_6 - 0.5L$$

When member one fails in tension,the safety margins of the other members are calculated below.

From the above table and appendix D, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



The corresponding safety margins can be written as:

$$M_{2\,(1)} = R_2$$

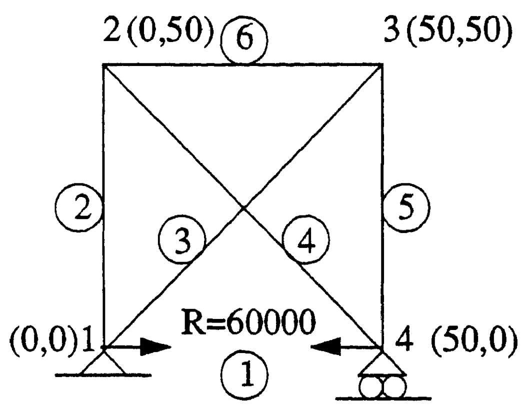
$$M_{3\,(1)} = R_3^{\ +} - S_3\,(tension) = R_3 - 113140 = R_3 - 1.4142L$$

$$M_{4(1)} = R_4$$

$$M_{5(1)} = R_5^- + S_5\,(compression) = R_5 - 80000 = R_5 - L$$

$$M_{6(1)} = R_6^- + S_6\,(compression) = R_6 - 80000 = R_6 - L$$

From the above table and appendix E, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{2(1)} = R_2^+ - S_2\,(tension) = R_2 - 60000 = R_2 - R_1$$

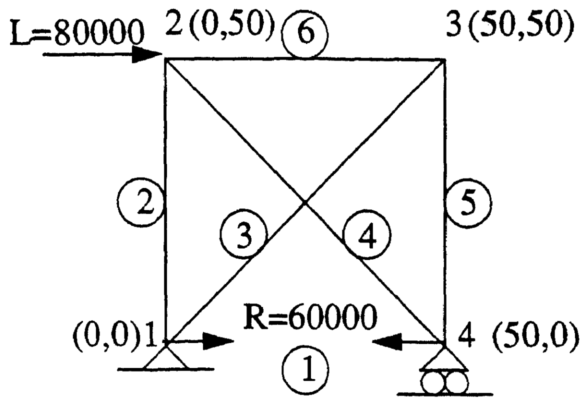
$$M_{3(1)} = R_3^- + S_3\,(compression) = R_3 - 84853 = R_3 - 1.4142R_1$$

$$M_{4(1)} = R_4^- + S_4\,(compression) = R_4 - 84853 = R_4 - 1.4142R_1$$

$$M_{5(1)} = R_5^+ - S_5\,(tension) = R_5 - 60000 = R_5 - R_1$$

$$M_{6(1)} = R_6^+ - S_6\,(tension) = R_6 - 60000 = R_6 - R_1$$

From the above table and appendix C1, one can refer to the force distributions.



Therefore, when member one fails in tension, the safety margins of the other members are as follows:

$$M_{2(1)} = R_2^{\ +} - S_2(tension) = R_2 - 60000 = R_2 - R_1$$

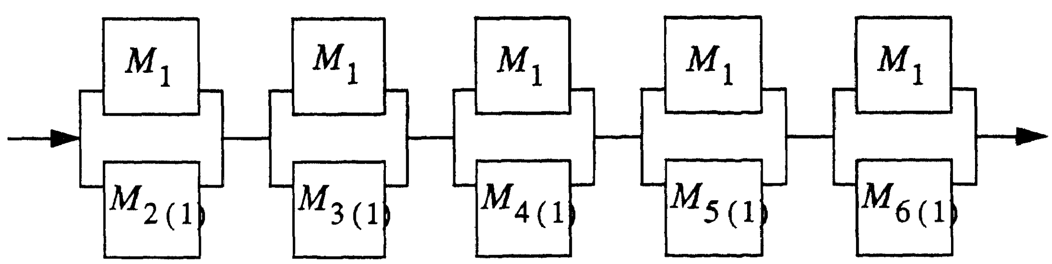
$$M_{3(1)} = R_3^{\ +} - S_3(tension) = R_3 - 28284 = R_3 + 1.4142R_1 - 1.4142L$$

$$M_{4(1)} = R_4^- + S_4(compression) = R_4 - 84853 = R_4 - 1.4142R_1$$

$$M_{5(1)} = R_5^- + S_5(compression) = R_5 - 20000 = R_5 + R_1 - L$$

$$M_{6(1)} = R_6^- + S_6(compression) = R_6 - 20000 = R_6 + R_1 - L$$

If any more member fails, *the stiffness mateix will become singular*; therefore, when member one failed, the failure modes of the structure can be plotted in a series system as shown:



The corresponding safety margins are as follows:

$$M_1 = R_1 - 0.5L$$

$$M_{2(1)} = R_2 - R_1$$

$$M_{3(1)} = R_3 + 1.4142R_1 - 1.4142L$$

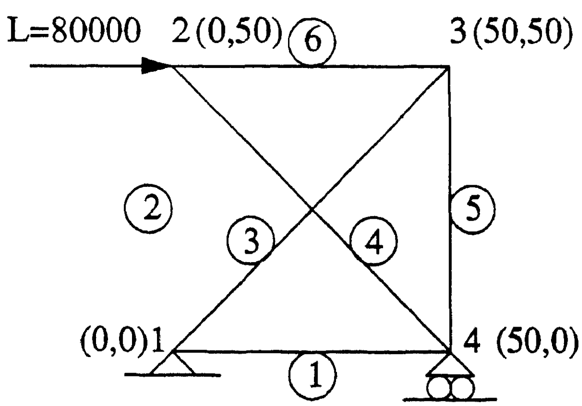
$$M_{4(1)} = R_4 - 1.4142R_1$$

$$M_{5(1)} = R_5 + R_1 - L$$

$$M_{6(1)} = R_6 + R_1 - L$$

When member two fails in tension, the safety margins of the other members are calculated below:

From the above table and appendix D1, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(2)} = R_1$$

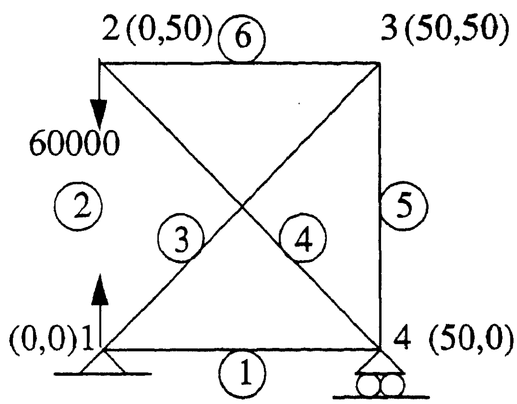
$$M_{3(2)} = R_3^{\,+} - S_3\,(tension) = R_3 - 113140 = R_3 - 1.4142L$$

$$M_{4(2)} = R_4$$

$$M_{5(2)} = R_5^- + S_5(compression) = R_5 - 80000 = R_5 - L$$

$$M_{6(2)} = R_6^- + S_6(compression) = R_6 - 80000 = R_6 - L$$

From the above table and appendix E1, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



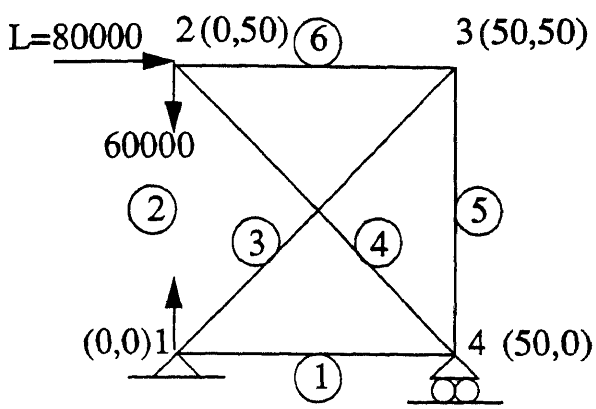
$$M_{3(2)} = R_3^- + S_3(compression) = R_3 - 84853 = R_3 - 1.4142R_2$$

$$M_{4(2)} = R_4^- + S_4(compression) = R_4 - 84853 = R_4 - 1.4142R_2$$

$$M_{5(2)} = R_5^+ - S_5(tension) = R_5 - 60000 = R_5 - R_2$$

$$M_{6(2)} = R_6^+ - S_6(tension) = R_6 - 60000 = R_6 - R_2$$

From the above table and appendix C2, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(2)} = R_1^{+} - S_1(t) = R_1 - 60000 = R_1 - R_2$$

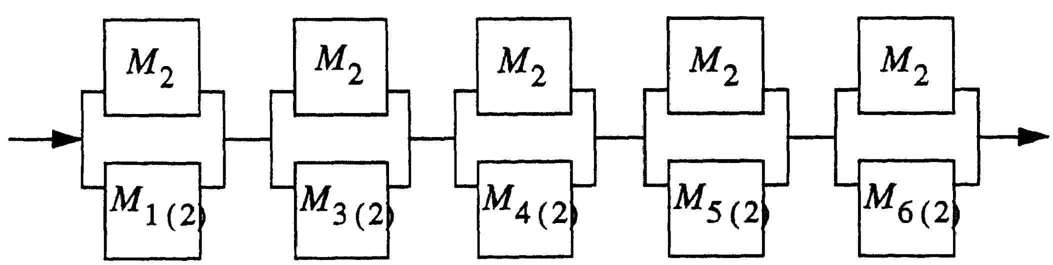
$$M_{3(2)} = R_2^{+} - S_2(t) = R_3 - 28284 = R_3 + 1.4142R_2 - 1.4142L$$

$$M_{4(2)} = R_4^{-} + S_4(c) = R_4 - 84853 = R_4 - 1.4142R_2$$

$$M_{5(2)} = R_5^{-} + S_5(c) = R_5 - 20000 = R_5 + R_2 - L$$

$$M_{6(2)} = R_6^{-} + S_6(c) = R_6 - 20000 = R_6 + R_2 - L$$

If any more member fails, *the stiffness matrix will become singular*; therefore, when member two failed, the failure modes of the structure can be plotted in a series system as follows:



The corresponding safety margins are as follows:

$$M_2 = R_2 - 0.5L$$

$$M_{1(2)} = R_1 - R_2$$

$$M_{3(2)} = R_3 + 1.4142R_2 - 1.4142L$$

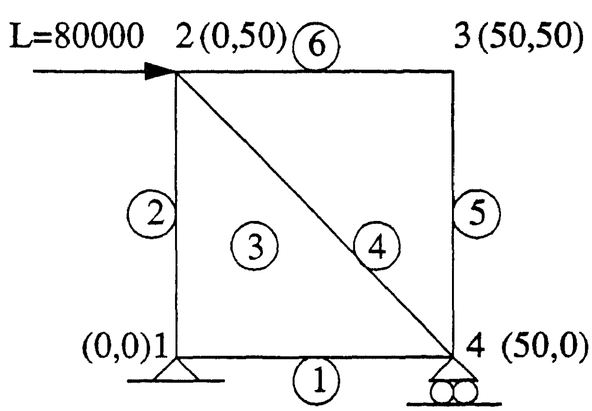
$$M_{4(2)} = R_4 - 1.4142R_2$$

$$M_{5(2)} = R_5 + R_2 - L$$

$$M_{6(2)} = R_6 + R_2 - L$$

When member three fails in tension, the safety margins of the other members are calculated below:

From the above table and appendix D2, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(3)} = R_1^{+} - S_1\,(tension) = R_1 - 80000 = R_1 - L$$

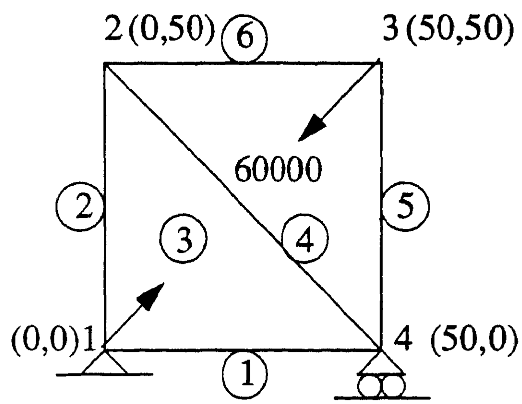
$$M_{2(3)} = R_2^{+} - S_2\,(tension) = R_2 - 80000 = R_2 - L$$

$$M_{4(3)} = R_4^{-} + S_4\,(compression) = R_4 - 113140 = R_4 - 1.4142L$$

$$M_{5\,(3)} = R_5$$

$$M_{6\,(3)} = R_6$$

From the above table and appendix E2, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1\,(3)} = R_1^- + S_1\,(compression) = R_1 - 42426 = R_1 - 0.7071R_3$$

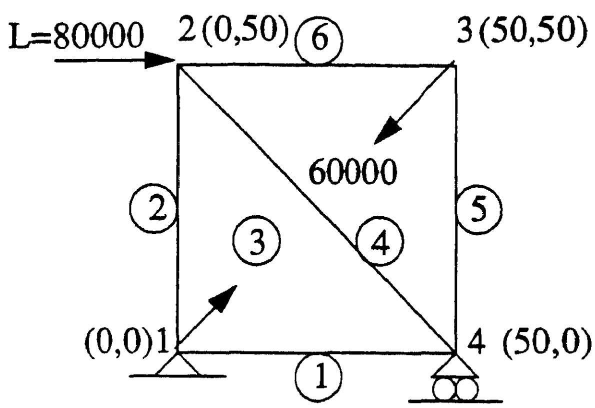
$$M_{2\,(3)} = R_2^- + S_2\,(compression) = R_2 - 42426 = R_2 - 0.7071R_3$$

$$M_{4\,(3)} = R_4^{\;+} - S_4\,(tension) = R_4 - 60000 = R_4 - R_3$$

$$M_{5\,(3)} = R_5^- + S_5\,(compression) = R_5 - 42426 = R_5 - 0.7071R_3$$

$$M_{6\,(3)} = R_6^- + S_6\,(compression) = R_6 - 42426 = R_6 - 0.7071R_3$$

From the above table and appendix C3, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(3)} = R_1^+ - S_1(tension) = R_1 - 37574 = R_1 + 0.7071R_3 - L$$

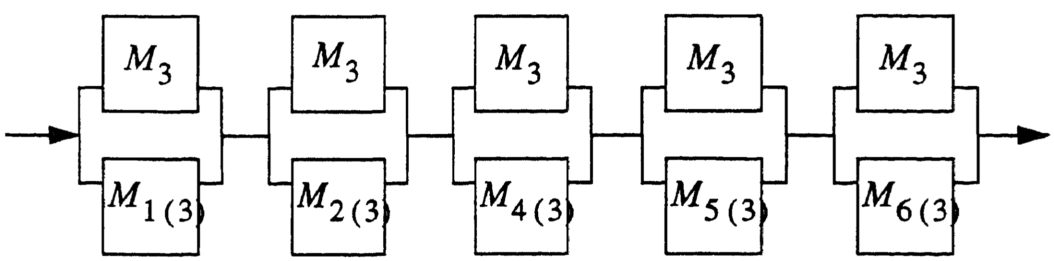
$$M_{2(3)} = R_2^+ - S_2(tension) = R_2 - 37574 = R_2 + 0.7071R_3 - L$$

$$M_{4(3)} = R_4^- + S_4(compression) = R_4 - 53137 = R_4 + R_3 - 1.4142L$$

$$M_{5(3)} = R_5^- + S_5(compression) = R_5 - 42426 = R_5 - 0.7071R_3$$

$$M_{6(3)} = R_6^- + S_6(compression) = R_6 - 42426 = R_6 - 0.7071R_3$$

If any more member fails, the stiffness matrix will become singular; therefore, when member three failed, the failure modes of the structure can be plotted in a series system as follows:



The corresponding safety margins are as follows:

$$M_3 = R_3 - 0.7071L$$

$$M_{1(3)} = R_1 + 0.7071R_3 - L$$

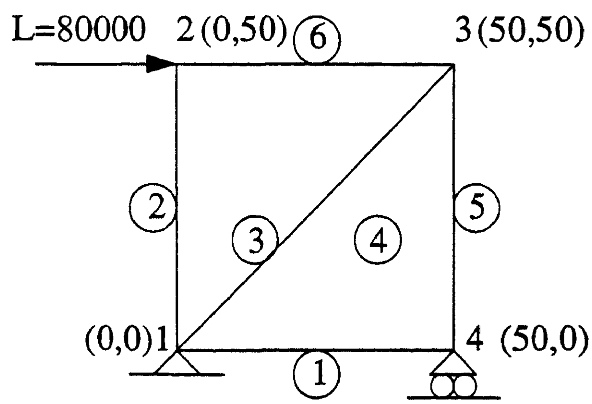
$$M_{2\,(3)} = R_2 + 0.7071 R_3 - L$$

$$M_{4\,(3)} = R_4 + R_3 - 1.4142L$$

$$M_{5\,(3)} = R_5 - 0.7071 R_3$$

$$M_{6\,(3)} = R_6 - 0.7071 R_3$$

When member four fails in compression, the safety margins of the other members are calculated below:



From the above table and appendix D3, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:

$$M_{1\,(4)} = R_1$$

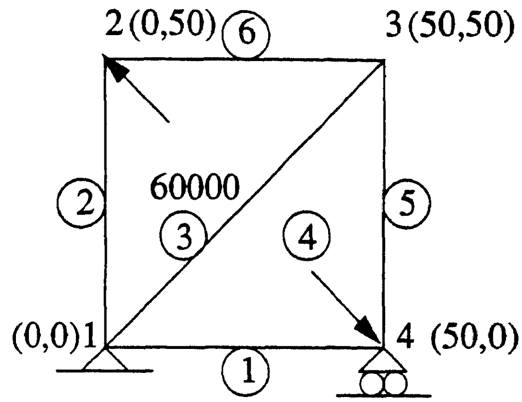
$$M_{2\,(4)} = R_2$$

$$M_{3\,(4)} = R_3^{+} - S_3\,(tension) = R_3 - 113140 = R_3 - 1.4142L$$

$$M_{5\,(4)} = R_5^{-} + S_5\,(compression) = R_5 - 80000 = R_5 - L$$

$$M_{6\,(4)} = R_6^{-} + S_6\,(compression) = R_6 - 80000 = R_6 - L$$

From the above table and appendix E3, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1\,(4)} = R_1^{\,+} - S_1\,(tension) = R_1 - 42426 = R_1 - 0.7071R_4$$

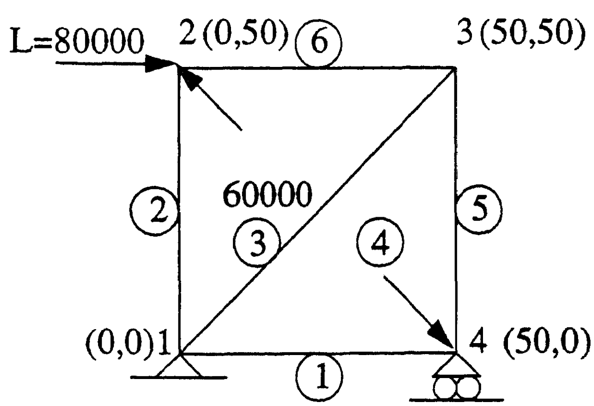
$$M_{2\,(4)} = R_2^{\,+} - S_2\,(tension) = R_2 - 42426 = R_2 - 0.7071R_4$$

$$M_{3\,(4)} = R_3^{-} + S_3\,(compression) = R_3 - 60000 = R_3 - R_4$$

$$M_{5\,(4)} = R_5^{\,+} - S_5\,(tension) = R_5 - 42426 = R_5 - 0.7071R_4$$

$$M_{6\,(4)} = R_6^{\,+} - S_6\,(tension) = R_6 - 42426 = R_6 - 0.7071R_4$$

From the above table and appendix C4, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(4)} = R_1^{+} - S_1(t) = R_1 - 42426 = R_1 - 0.7071R_4$$

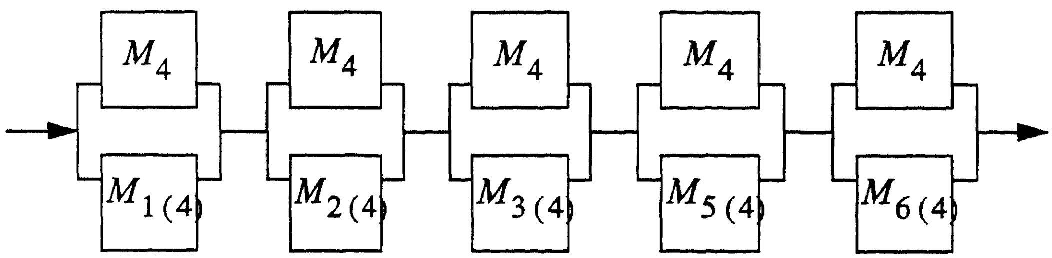
$$M_{2(4)} = R_2^{+} - S_2(t) = R_2 - 42426 = R_2 - 0.7071R_4$$

$$M_{3(4)} = R_3^{+} - S_3(t) = R_3 - 53137 = R_3 + R_4 - 1.4142L$$

$$M_{5(4)} = R_5^{-} + S_5(c) = R_5 - 37574 = R_5 + 0.7071R_4 - L$$

$$M_{6(4)} = R_6^{-} + S_6(c) = R_6 - 37574 = R_6 + 0.7071R_4 - L$$

If any more member fails, the stiffness matrix will become singular; therefore, when member four failed, the failure modes of the structure can be plotted in a series system as shown:



The corresponding safety margins are as follows:

$$M_4 = R_4 - 0.7071L$$

$$M_{1(4)} = R_1 - 0.7071R_4$$

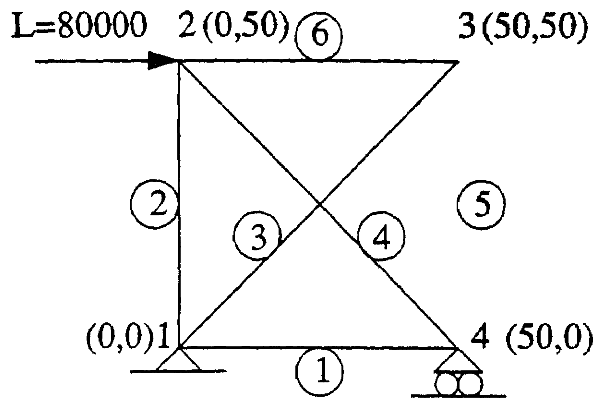
$$M_{2(4)} = R_2 - 0.7071R_4$$

$$M_{3(4)} = R_3 + R_4 - 1.4142L$$

$$M_{5(4)} = R_5 + 0.7071R_4 - L$$

$$M_{6(4)} = R_6 + 0.7071R_4 - L$$

When member five fails in compression, the safety margins of the other members are calculated below:



From the above table and appendix D4, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:

$$M_{1\,(5)} = R_1^{\;+} - S_1\,(tension) = R_1 - L$$

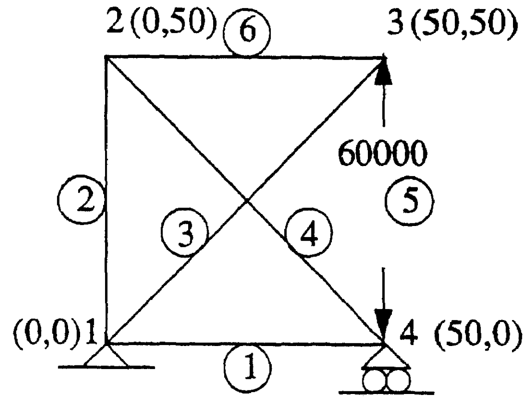
$$M_{2\,(5)} = R_2^{\;+} - S_2\,(tension) = R_2 - L$$

$$M_{3\,(5)} = R_3$$

$$M_{4\,(5)} = R_4^- + S_4\,(compression) = R_4 - 113140 = R_4 - 1.4142L$$

$$M_{6\,(5)} = R_6$$

From the above table and appendix E4, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(5)} = R_1^- + S_1\,(compression) = R_1 - 60000 = R_1 - R_5$$

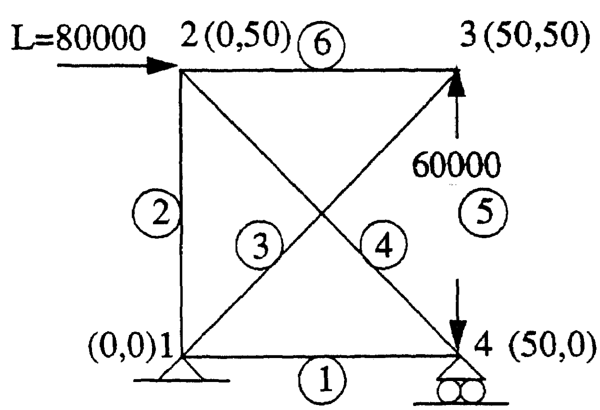
$$M_{2(5)} = R_2^- + S_2\,(compression) = R_2 - 60000 = R_2 - R_5$$

$$M_{3(5)} = R_3^{\,+} - S_3\,(tension) = R_3 - 84853 = R_3 - 1.4142R_5$$

$$M_{4(5)} = R_4^{\,+} - S_4\,(tension) = R_4 - 84853 = R_4 - 1.4142R_5$$

$$M_{6(5)} = R_6^- + S_6\,(compression) = R_6 - 60000 = R_6 - R_5$$

From the above table and appendix C5, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(5)} = R_1^{\,+} - S_1\,(t) = R_1 - 20000 = R_1 + R_5 - L$$

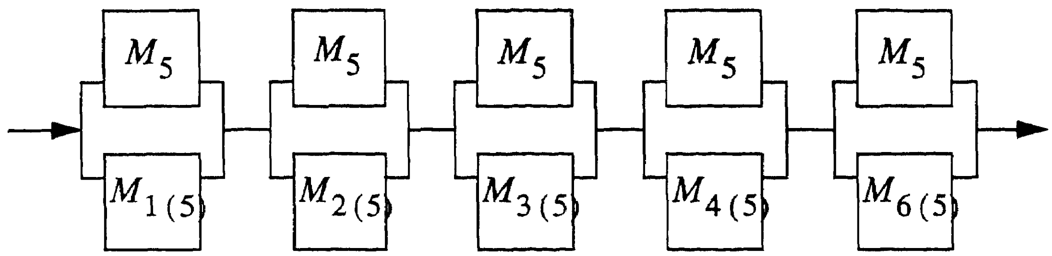
$$M_{2(5)} = R_2^{+} - S_2(t) = R_2 - 20000 = R_2 + R_5 - L$$

$$M_{3(5)} = R_3^{+} - S_3(t) = R_3 - 84853 = R_3 - 1.4142R_5$$

$$M_{4(5)} = R_4^{-} + S_4(c) = R_4 - 28284 = R_4 + 1.4142R_5 - 1.4142L$$

$$M_{6(5)} = R_6^{-} + S_6(c) = R_6 - 60000 = R_6 - R_5$$

If any more member fails, the stiffness matrix will become singular; therefore, when member five failed, the failure modes of the structure can be plotted in a series as shown:



The corresponding safety margins are as follows:

$$M_5 = R_5 - 0.5L; M_{1(5)} = R_1 + R_5 - L$$

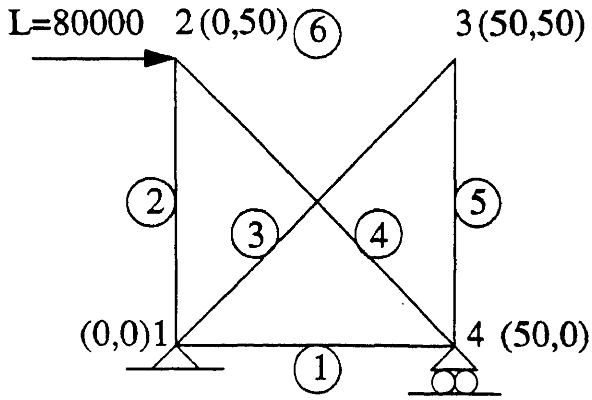
$$M_{2(5)} = R_2 + R_5 - L$$

$$M_{3(5)} = R_3 - 1.4142R_5$$

$$M_{4(5)} = R_4 + 1.4142R_5 - 1.4142L$$

$$M_{6(5)} = R_6 - R_5$$

When member six fails in compression, the safety margins of the other members are calculated below:



From the above table and appendix D5, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:

$$M_{1\,(6)} = R_1^{+} - S_1\,(tension) = R_1 - 80000 = R_1 - L$$

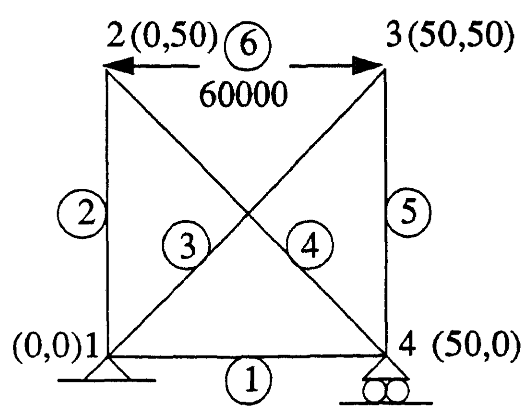
$$M_{2\,(6)} = R_2^{+} - S_2\,(tension) = R_2 - 80000 = R_2 - L$$

$$M_{3\,(6)} = R_3$$

$$M_{4\,(6)} = R_4^{-} + S_4\,(compression) = R_4 - 113140 = R_4 - 1.4142L$$

$$M_{5\,(6)} = R_5$$

From the above table and appendix E5, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(6)} = R_1^- + S_1\,(compression) = R_1 - 60000 = R_1 - R_6$$

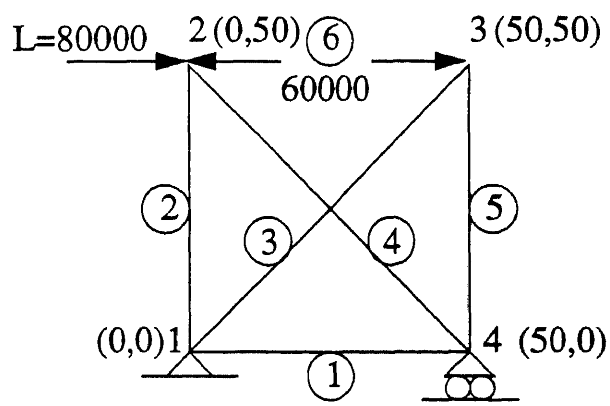
$$M_{2(6)} = R_2^- + S_2\,(compression) = R_2 - 60000 = R_2 - R_6$$

$$M_{3(6)} = R_3^+ - S_3\,(tension) = R_3 - 84853 = R_3 - 1.4142R_6$$

$$M_{4(6)} = R_4^+ - S_4\,(tension) = R_4 - 84853 = R_4 - 1.4142R_6$$

$$M_{5(6)} = R_5^- + S_5\,(compression) = R_5 - 60000 = R_5 - R_6$$

From the above table and appendix C6, one can refer to the force distributions. Therefore, the safety margins of the individual bars are given by:



$$M_{1(6)} = R_1^+ - S_1\,(t) = R_1 - 20000 = R_1 + R_6 - L$$

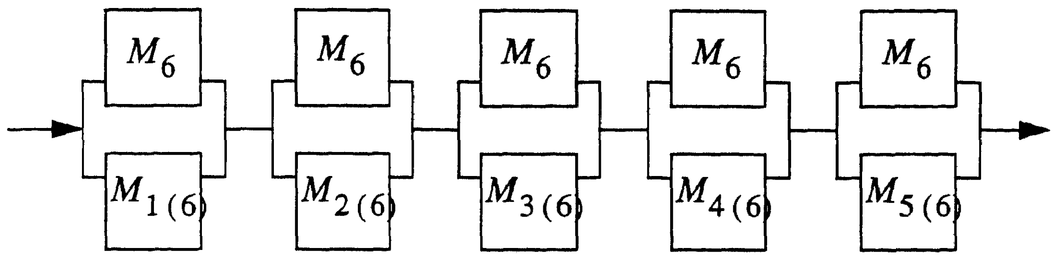
$$M_{2(6)} = R_2^+ - S_2\,(t) = R_2 - 20000 = R_2 + R_6 - L$$

$$M_{3(6)} = R_3^+ - S_3\,(t) = R_3 - 84853 = R_3 - 1.4142R_6$$

$$M_{4(6)} = R_4^- + S_4\,(c) = R_4 - 28284 = R_4 + 1.4142R_6 - 1.4142L$$

$$M_{5(6)} = R_5^- + S_5\,(c) = R_5 - 60000 = R_5 - R_6$$

If any more member fails, the stiffness matrix will become singular; therefore, when member six failed, the failure modes of the structure can be plotted in a series system as shown:



The corresponding safety margins are as follows:

$$M_6 = R_6 - 0.5L$$

$$M_{1(6)} = R_1 + R_6 - L$$

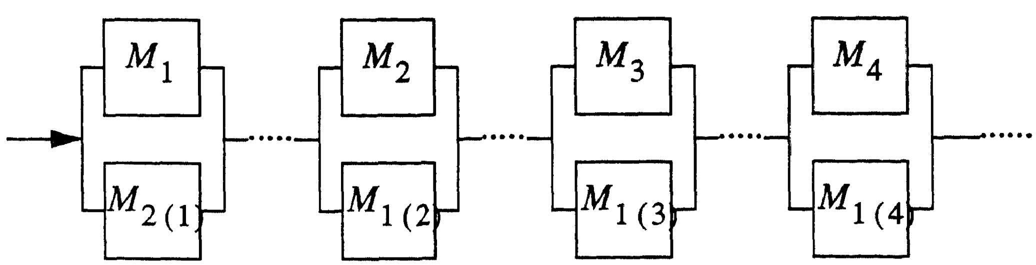
$$M_{2(6)} = R_2 + R_6 - L$$

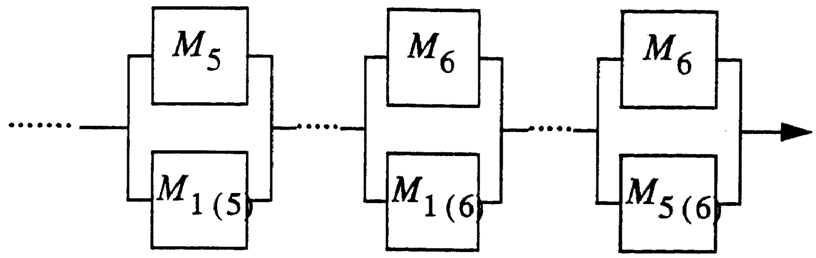
$$M_{3(6)} = R_3 - 1.4142R_6$$

$$M_{4(6)} = R_4 + 1.4142R_6 - 1.4142L$$

$$M_{5(6)} = R_5 - R_6$$

If one considers the probability of failure of the whole structure, there are thirty different possibilities (failure modes) which can cause the structure failure as shown below:




M_5
$M_{1(5)}$
M_6
$M_{1(6)}$
M_6
$M_{5(6)}$

CHAPTER 9
STRUCTURAL RELIABILITY ANALYSIS USING ANSYS

9.1 INTRODUCTION

For some structures the reliabilities of structures are calculated on the basis of failure of single components, where the probability of failure of any component and the correlation among failed components are taken into account. Then combine all the failure components to make up the series system.The modelling of this type is called system reliability at level 1. The evaluation of the structural reliability can be obtained with satisfactory accuracy by only including failure components with high probabilities of failure. Such significant failure components can be selected by choosing those failure components with P_f values in an interval $[P_{fmax}, P_{fmax} - \Delta P_f]$, where ΔP_f must be chosen properly.

For some structures the reliabilities of the structures are calculated on the basis of failure of two failure components, where the probability of failure of any pair of failure components and the correlation among failure pairs are taken into account, and then all the failure pairs are combined to make up the series system. The modelling of this type is called system reliability at level 2. The evaluation of the structural reliability can be obtained with satisfactory accuracy by only including failure pairs with high probability of failure.

To obtain the so-called significant pairs of failure components, the structure is modified by assuming failure in the significant failure components and applying artificial loads which are the strength capacities of the failure components if the components are ductile.

No artificial loads are applied if the failure components are brittle. Then the modified structure is analysed elastically and new P_f values are calculated for all surviving components. Surviving components with high P_f values are then combined with the significant failure components so that the significant pairs of failure components can be determined. By continuing in the same procedure, system reliability at level N, N=3,4,5...... can be defined.

The most frequently used modelling of system reliability is system reliability at the mechanism level. Usually the number of mechanisms (failure modes) is very large; therefore, only some reasonable number of significant mechanisms should be considered. The procedure described above can be continued until formations of mechanisms, but sometimes such a procedure will be very inconvenient due to many reanalyses needed for a highly redundant structure. Therefore, it is better to base the ANSYS reliability analysis on the fundamental mechanisms and on the linear combinations of fundamental mechanisms. (see the platic theory of structures for details).

9.2 TRANSFORMATIONS OF NON-NORMAL BASIC VARIABLES

In general basic variables can not be modelled by a normal distribution; In such a case the transformation from the non-normal distribution to a normal distribution is needed. If all basic variables are normally distributed, the evaluation of the structural reliability will be greatly simplified; The only information needed will then be the expected values, the standard deviations and the correlation coefficients.

Many different transformation methods have been suggested to overcome the problem. A very accurate method suggested by *Rackwitz and Fiessler* is discussed below: A non-normal variable is transformed into a normal variable so that at the design point the corresponding density functions and the distribution functions are same , i.e.:

$$F_{X_i}(x''_i) = \Phi\left(\frac{x''_i - \mu'_{X_i}}{\sigma'_{X_i}}\right) \text{.......(1)}$$

$$f_{X_i}(x''_i) = \frac{1}{\sigma'_{X_i}}\varphi\left(\frac{x''_i - \mu'_{X_i}}{\sigma'_{X_i}}\right) \text{.......(2)}$$

where $\bar{x}'' = (x''_1, \ldots, x''_i, \ldots, x''_n)$ is the design point, μ'_{X_i} and σ'_{X_i} are the mean value and standard deviation of the transformed normal distribution.

From equation (1) , (2) the mean value and the standard deviation of the transformed normal distribution can be given by:

$$\mu'_{X_i} = x''_i - \Phi^{-1}(F_{X_i}(x''_i))\,\sigma'_{X_i}$$

$$\sigma'_{X_i} = \frac{\varphi(\Phi^{-1}(F_{X_i}(x''_i)))}{f_{X_i}(x''_i)}$$

where the design point $\bar{x}''$ can be obtained by iterative method.

A simpler way called *the multiplication factor method* suggested by the *Thoft-Christensen* to define the design point is as follows:

$$s'' = \mu_S + \alpha_S \sigma_S$$

where μ_S and σ_S are the orginal mean value and standard deviation of non- normal distribution, and where α_S is a positive multiplication factor, and it can be determined by experiment.

Therefore, at design point s''

$$F_S(s'') = \Phi\left(\frac{s'' - \mu'_S}{\sigma'_S}\right)$$

$$f_S(s'') = \frac{1}{\sigma'_S}\varphi\left(\frac{s'' - \mu'_S}{\sigma'_S}\right)$$

Finally, the unknown equivalent mean value and standard deviation can be given by:

$$\mu'_S = s'' - \Phi^{-1}(F_S(s''))\,\sigma'_S$$

$$\sigma'_S = \frac{\varphi(\Phi^{-1}(F_S(s'')))}{f_S(s'')}$$

According to experience, in some cases one can use the original mean value μ_S and original standard deviation σ_S of a non-normal distribution variable instead of the equivalent mean value μ'_S and equivalent standard deviation σ'_S of the approximate normal distribution variable.

9.3ESTIMATE OF STRUCTURAL RELIABILITY

In this section it is assumed that all basic variables are normally distributed. The structure is considered at a fixed point in time so that only static situation is discussed and all elasticity coefficients are assumed deterministic.

Example 1. Consider again the structure of example 1 of chapter 8. The expected values and standard deviations are listed below. Assume that the bars at the same level are perfectly correlated; otherwise, uncorrelated.

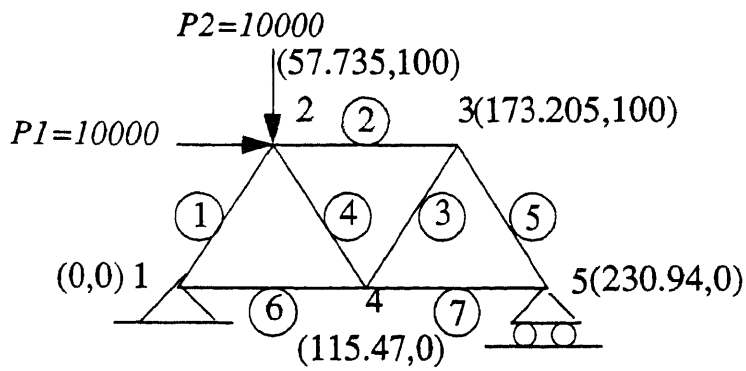


figure a

$$\mu_{p_1} = 10000, \mu_{p_2} = 10000, \mu_R = 60000$$

$$\sigma_{p_1} = 1000, \sigma_{p_1} = 1000, \sigma_R = 6000$$

The safety margins of the structure are given as follows:

$$M_1 = R_1 + 0.5P_1 - 0.866P_2$$

$$M_2 = R_2 - 0.5P_1 - 0.289P_2$$

$$M_3 = R_3 - 0.5P_1 - 0.289P_2$$

$$M_4 = R_4 - 0.5P_1 - 0.289P_2$$

$$M_5 = R_5 - 0.5P_1 - 0.289P_2$$

$$M_6 = R_6 - 0.75P_1 - 0.433P_2$$

$$M_7 = R_7 - 0.25P_1 - 0.144P_2$$

Due to the perfect correlated at the same level the safety margins can be reduced as follows:

$$M_1 = R_1 + 0.5P_1 - 0.866P_2$$

$$M_2 = R_2 - 0.5P_1 - 0.289P_2$$

$$M_3 = R_1 - 0.5P_1 - 0.289P_2$$

$$M_4 = R_6 - 0.75P_1 - 0.433P_2$$

$$M_5 = R_6 - 0.25P_1 - 0.144P_2$$

The random variables are normalized by

$$Z_1 = \frac{P_1 - \mu_{P_1}}{\sigma_{P_1}}, P_1 = \sigma_{P_1} Z_1 + \mu_{P_1} = 1000Z_1 + 10000$$

$$Z_2 = \frac{P_2 - \mu_{P_2}}{\sigma_{P_2}}, P_2 = \sigma_{P_2} Z_2 + \mu_{P_2} = 1000Z_2 + 10000$$

$$Z_3 = \frac{R_1 - \mu_R}{\sigma_R}, R_1 = \sigma_R Z_3 + \mu_R = 6000Z_3 + 60000$$

$$Z_4 = \frac{R_2 - \mu_R}{\sigma_R}, R_2 = \sigma_R Z_4 + \mu_R = 6000Z_4 + 60000$$

$$Z_5 = \frac{R_6 - \mu_R}{\sigma_R}, R_6 = \sigma_R Z_5 + \mu_R = 6000Z_5 + 60000$$

By the substitutions, one can get:

$$M_1 = R_1 + 0.5P_1 - 0.866P_2 = (6000Z_3 + 60000) + 0.5\,(1000Z_1 + 10000)$$
$$-0.866\,(1000Z_2 + 10000)$$

$$M_1 = 56340 + 500Z_1 - 866Z_2 + 6000Z_3$$

$$M_1 = 9.26 + 0.0822Z_1 - 0.1424Z_2 + 0.9864Z_3 \text{ (after unification)}$$

$$\Phi\,(-9.26) = 0.1098\times10^{-19}$$

$$M_2 = R_2 - 0.5P_1 - 0.289P_2 = (6000Z_4 + 60000) - 0.5\,(1000Z_1 + 10000)$$
$$-0.289\,(1000Z_2 + 10000)$$

$$M_2 = 52110 - 500Z_1 - 289Z_2 + 6000Z_4$$

$$M_2 = 8.65 - 0.0829Z_1 - 0.0479Z_2 + 0.9954Z_4$$

$$\Phi\,(-8.65) = 0.2736\times10^{-17}$$

$$M_3 = R_1 - 0.5P_1 - 0.289P_2 = (6000Z_3 + 60000) - 0.5\,(1000Z_1 + 10000)$$
$$-0.289\,(1000Z_2 + 10000)$$

$$M_3 = 52110 - 500Z_1 - 289Z_2 + 6000Z_3$$

$$M_3 = 8.65 - 0.0829Z_1 - 0.0479Z_2 + 0.9954Z_3$$

$$\Phi\,(-8.65) = 0.2736\times10^{-17}$$

$$M_4 = R_6 - 0.75P_1 - 0.433P_2 = (6000Z_5 + 60000) - 0.75\,(1000Z_1 + 10000)$$
$$-0.433\,(1000Z_2 + 10000)$$

$$M_4 = 48170 - 750Z_1 - 433Z_2 + 6000Z_5$$

$$M_4 = 7.95 - 0.1237Z_1 - 0.0714Z_2 + 0.9897Z_5$$

$$\Phi\,(-7.95) = 0.9823\times10^{-15}$$

$$M_5 = R_6 - 0.25P_1 - 0.144P_2 = (6000Z_5 + 60000) - 0.25\,(1000Z_1 + 10000)$$
$$-0.144\,(1000Z_2 + 10000)$$

$$M_5 = 56060 - 250Z_1 - 144Z_2 + 6000Z_5$$

$$M_5 = 9.33 - 0.0416Z_1 - 0.024Z_2 + 0.9988Z_5$$

$$\Phi\,(-9.33) = 0.553\times10^{-20}$$

The coefficients of correlation are

$$\rho_{12} = -0.0822 \times 0.0822 + 0.1424 \times 0.0479 = 0.00000658 \approx 0$$

$$\rho_{13} = -0.0822 \times 0.0829 + 0.1424 \times 0.0479 + 0.9864 \times 0.9954 = 0.9819$$

$$\rho_{14} = -0.0822 \times 0.1237 + 0.1424 \times 0.0714 = -0.00000078 \approx 0$$

$$\rho_{15} = -0.0822 \times 0.0416 + 0.1424 \times 0.024 = -0.00000192 \approx 0$$

$$\rho_{23} = 0.0829 \times 0.0829 + 0.0479 \times 0.0479 = 0.0092$$

$$\rho_{24} = 0.0829 \times 0.1237 + 0.0479 \times 0.0714 = 0.0137$$

$$\rho_{25} = 0.0829 \times 0.0416 + 0.0479 \times 0.024 = 0.0046$$

$$\rho_{34} = 0.0829 \times 0.1237 + 0.0479 \times 0.0714 = 0.0137$$

$$\rho_{35} = 0.0829 \times 0.0416 + 0.0479 \times 0.024 = 0.0046$$

$$\rho_{45} = 0.1237 \times 0.0416 + 0.0714 \times 0.024 + 0.9897 \times 0.9988 = 0.9953$$

$$\Phi(-7.95) = 0.9823 \times 10^{-15} > \Phi(-8.65) = 0.2736 \times 10^{-17} =$$

$$\Phi(-8.65) = 0.2736 \times 10^{-17} > \Phi(-9.26) = 0.1098 \times 10^{-19} >$$

$$\Phi(-9.33) = 0.553 \times 10^{-20}$$

$$\Phi_2(-\beta_2, -\beta_1; \rho) = \Phi_2(-8.65, -7.95; 0.0137) = 6.534 \times 10^{-33}$$

$$\Phi_2(-\beta_3, -\beta_1; \rho) = \Phi_2(-8.65, -7.95; 0.0137) = 6.534 \times 10^{-33}$$

$$\Phi_2(-\beta_3,-\beta_2;\rho) = \Phi_2(-8.65,-8.65;0.092) = 5.472\times10^{-18}$$

$$\Phi_2(-\beta_4,-\beta_1;\rho) = \Phi_2(-9.26,-7.95;0) = 1.0786\times10^{-35}$$

$$\Phi_2(-\beta_4,-\beta_2;\rho) = \Phi_2(-9.26,-8.65;0) = 3.004\times10^{-38}$$

$$\Phi_2(-\beta_4,-\beta_3;\rho) = \Phi_2(-9.26,-8.65;0.9819) = 1.04\times10^{-20}$$

$$\Phi_2(-\beta_5,-\beta_1;\rho) = \Phi_2(-9.33,-7.95;0.9953) = 5.311\times10^{-21}$$

$$\Phi_2(-\beta_5,-\beta_2;\rho) = \Phi_2(-9.33,-8.65;0.0046) = 2.14\times10^{-38}$$

$$\Phi_2(-\beta_5,-\beta_3;\rho) = \Phi_2(-9.33,-8.65;0.0046) = 2.14\times10^{-38}$$

$$\Phi_2(-\beta_5,-\beta_4;\rho) = \Phi_2(-9.33,-9.26;0) = 6.0719\times10^{-41}$$

By using the Ditlevsen bounds:

upper bound: $P_f \le \sum_{i=1}^{n}\Phi(-\beta_i) - \sum_{i=2,\,j<i}^{n} max\Phi_2(-\beta_i,-\beta_j;\rho)$

$$\begin{aligned}
P_f \le\ & 9.878\times10^{-16} - max[\Phi_2(-\beta_2,-\beta_1;\rho)] \\
& -max[\Phi_2(-\beta_3,-\beta_2;\rho),\Phi_2(-\beta_3,-\beta_1;\rho)] \\
& -max[\Phi(-\beta_4,-\beta_1;\rho),\Phi(-\beta_4,-\beta_2;\rho),\Phi(-\beta_4,-\beta_3;\rho)] \\
& -max[\Phi(-\beta_5,-\beta_1;\rho),\Phi(-\beta_5,-\beta_2;\rho),\Phi(-\beta_5,-\beta_3;\rho),\Phi(-\beta_5,-\beta_4;\rho)]
\end{aligned}$$

$$= 9.878\times10^{-16} - 6.534\times10^{-33} - 5.472\times10^{-18} - 1.04\times10^{-20} - 5.311\times10^{-21}$$

$$= 9.823\times10^{-16}$$

lower bound: $P_f \geq \Phi(-\beta_1) + \sum_{i=2}^{n} max\left[\Phi(-\beta_i) - \sum_{j=1}^{i-1} \Phi_2(-\beta_i, -\beta_j;\rho), 0\right]$

$$P_f \geq 9.823\times10^{-16} + max[\Phi(-\beta_2) - \Phi_2(-\beta_2, -\beta_1;\rho), 0]$$

$$+ max[\Phi(-\beta_3) - \Phi_2(-\beta_3, -\beta_1;\rho) - \Phi_2(-\beta_3, -\beta_2;\rho), 0] +$$

$$max[\Phi(-\beta_4) - \Phi_2(-\beta_4, -\beta_1;\rho) - \Phi_2(-\beta_4, -\beta_2;\rho) - \Phi_2(-\beta_4, -\beta_3;\rho), 0]$$

$$+$$

$$max[\Phi(-\beta_5) - \Phi_2(-\beta_5, -\beta_1;\rho) - \Phi_2(-\beta_5, -\beta_2;\rho) - \Phi_2(-\beta_5, -\beta_3;\rho)$$

$$-\Phi_2(-\beta_5, -\beta_4;\rho), 0]$$

$$P_f \geq 9.823\times10^{-16} + 0.2736\times10^{-17} - 6.534\times10^{-33} + 0.2736\times10^{-17}$$

$$- 6.534\times10^{-33} - 5.472\times10^{-18} + 0.1098\times10^{-19} - 1.0786\times10^{-35} - 3.004\times10^{-38}$$

$$- 1.04\times10^{-20} + 0.533\times10^{-20} - 5.311\times10^{-21} - 2.14\times10^{-38} - 2.14\times10^{-38}$$

$$-6.0719\times10^{-41} = 9.823\times10^{-16}$$

Therefore, one can get the probability of failure and probability index of the structure as follows:

$P_f = 9.823\times10^{-16}$ and $\beta = -\Phi^{-1}(P_f) = -7.95$

From the result one can expect that the failure bar six is the only dominant failure mode.

$$M_6 = R_6 - 0.75P_1 - 0.433P_2$$

When $\mu_{p_1} = 30000, \mu_{p_2} = 30000, \mu_R = 60000, \sigma_{p_1} = 3000, \sigma_{p_1} = 3000,$
$\sigma_R = 6000$

The structure reliability can be evaluated as follows:

$$Z_1 = \frac{R - \mu_R}{\sigma_R}, \; R = \sigma_R Z_1 + \mu_R = 6000Z_1 + 60000$$

$$Z_2 = \frac{P_1 - \mu_{P_1}}{\sigma_{P_1}}, P_1 = \sigma_{P_1} Z_2 + \mu_{P_1} = 3000Z_2 + 30000$$

$$Z_3 = \frac{P_2 - \mu_{P_2}}{\sigma_{P_2}}, P_2 = \sigma_{P_2} Z_3 + \mu_{P_2} = 3000Z_3 + 30000$$

$$M_6 = R_6 - 0.75P_1 - 0.433P_2 = (6000Z_1 + 60000) - 0.75\,(3000Z_2 + 30000)$$
$$-0.433\,(3000Z_3 + 30000)$$

$$M = 24510 + 6000Z_1 - 2250Z_2 - 1299Z_3$$

$$M = 3.75 + 0.9177Z_1 - 0.3441Z_2 - 0.1987Z_3 \text{ (after unification)}$$

$$P_f = \Phi(-3.75) = 8.837 \times 10^{-5}$$

when $\mu_{p_1} = 50000, \mu_{p_2} = 50000, \mu_R = 60000, \sigma_{p_1} = 5000, \sigma_{p_2} = 5000$

$\sigma_R = 6000$

The Structure reliability can be evaluated as follows:

$$Z_1 = \frac{R - \mu_R}{\sigma_R}\ , \ R = \sigma_R Z_1 + \mu_R = 6000Z_1 + 60000$$

$$Z_2 = \frac{P_1 - \mu_{P_1}}{\sigma_{P_1}}, P_1 = \sigma_{P_1} Z_2 + \mu_{P_1} = 5000Z_2 + 50000$$

$$Z_3 = \frac{P_2 - \mu_{P_2}}{\sigma_{P_2}}, P_2 = \sigma_{P_2} Z_3 + \mu_{P_2} = 5000Z_3 + 50000$$

$$M = R - 0.75P_1 - 0.433P_2 = (6000Z_1 + 60000) - 0.75\,(5000Z_2 + 50000)$$

$$-0.433\,(5000Z_3 + 50000)$$

$$M = 850 + 6000Z_1 - 3750Z_2 - 2165Z_3$$

$$M = 0.11 + 0.8109Z_1 - 0.5068Z_2 - 0.2926Z_3$$

$$P_f = \Phi(-0.11) = 0.4562$$

when $\mu_{p_1} = 20000, \mu_{p_2} = 20000, \mu_R = 60000, \sigma_{p_1} = 2000, \sigma_{p_2} = 2000$

$\sigma_R = 6000$

The Structure reliability can be evaluated as follows:

$$Z_1 = \frac{R - \mu_R}{\sigma_R}, \; R = \sigma_R Z_1 + \mu_R = 6000Z_1 + 60000$$

$$Z_2 = \frac{P_1 - \mu_{P_1}}{\sigma_{P_1}}, P_1 = \sigma_{P_1} Z_2 + \mu_{P_1} = 2000Z_2 + 20000$$

$$Z_3 = \frac{P_2 - \mu_{P_2}}{\sigma_{P_2}}, P_2 = \sigma_{P_2} Z_3 + \mu_{P_2} = 2000Z_3 + 20000$$

$$M = R - 0.75P_1 - 0.433P_2 = (6000Z_1 + 60000) - 0.75\,(2000Z_2 + 20000)$$

$$-0.433\,(2000Z_3 + 20000)$$

$$M = 36340 + 6000Z_1 - 1500Z_2 - 866Z_3$$

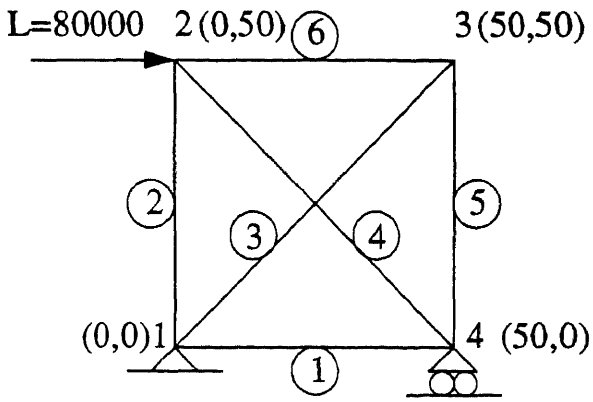
$$M = 5.82 + 0.9608Z_1 - 0.2402Z_2 - 0.1387Z_3$$

$$P_f = \Phi(-5.82) = 2.951\times10^{-9}$$

Example 2. Consider the structure of example 3 of chapter 8. Calculate the reliability of the structure. The expected values and standard deviations are given below. Assume that:

$R_i^{+} = R_i^{-} = 60000$, $(i = 1, 2, ..., 6)$ are independent.

$$\mu_L = 80000, \mu_R = 60000, \sigma_L = 8000, \sigma_R = 6000$$



$$Z_i = \frac{X_i - \mu_{X_i}}{\sigma_{X_i}}, X_i = \sigma_{X_i} Z_i + \mu_{X_i}$$

$$R_i = \sigma_{R_i} Z_i + \mu_{R_i} = 6000 Z_i + 60000, L = \sigma_L Z_7 + \mu_L = 8000 Z_7 + 80000$$

The safety margins of the structure are given as:

$$M_1 = R_1 - 0.5L = 2.77 + 0.8321 Z_1 - 0.5547 Z_7$$

$$M_{2(1)} = R_2 - R_1 = 0 - 0.7071 Z_1 + 0.7071 Z_7$$

$$M_{3(1)} = R_3 + 1.414 R_1 - 1.414 L = 2.06 + 0.5523 Z_1 + 0.3906 Z_3 - 0.7365 Z_7$$

$$M_{4(1)} = R_4 - 1.4142 R_1 = -2.39 - 0.8165 Z_1 + 0.5774 Z_4$$

$$M_{5\,(1)} = R_5 + R_1 - L = 3.43 + 0.5145Z_1 + 0.5145Z_5 - 0.686Z_7$$

$$M_{6\,(1)} = R_6 + R_1 - L = 3.43 + 0.5145Z_1 + 0.5145Z_6 - 0.686Z_7$$

$$M_2 = R_2 - 0.5L = 2.77 + 0.8321Z_2 - 0.5547Z_7$$

$$M_{1\,(2)} = R_1 - R_2 = 0 - 0.7071Z_1 + 0.7071Z_2$$

$$M_{3\,(2)} = R_3 + 1.414R_2 - 1.4142L = 2.06 + 0.5523Z_2 + 0.3906Z_3 - 0.7365Z_7$$

$$M_{4\,(2)} = R_4 - 1.4142R_2 = -2.39 - 0.8165Z_2 + 0.5774Z_4$$

$$M_{5\,(2)} = R_5 + R_2 - L = 3.43 + 0.5145Z_2 + 0.5145Z_5 - 0.686Z_7$$

$$M_{6\,(2)} = R_6 + R_2 - L = 3.43 + 0.5145Z_2 + 0.5145Z_6 - 0.686Z_7$$

$$M_3 = R_3 - 0.7071L = 0.42 + 0.7276Z_3 - 0.686Z_7$$

$$M_{1\,(3)} = R_1 + 0.7071R_3 - L = 2.06 + 0.5523Z_1 + 0.3906Z_3 - 0.7365Z_7$$

$$M_{2\,(3)} = R_2 + 0.7071R_3 - L = 2.06 + 0.5523Z_2 + 0.3906Z_3 - 0.7365Z_7$$

$$M_{4\,(3)} = R_4 + R_3 - 1.4142L = 0.49 + 0.4243Z_3 + 0.4243Z_4 - 0.8Z_7$$

$$M_{5\,(3)} = R_5 - 0.7071R_3 = 2.39 - 0.5773Z_3 + 0.8165Z_5$$

$$M_{6(3)} = R_6 - 0.7071R_3 = 2.39 - 0.5773Z_3 + 0.8165Z_6$$

$$M_4 = R_4 - 0.7071L = 0.42 + 0.7276Z_4 - 0.686Z_7$$

$$M_{1(4)} = R_1 - 0.7071R_4 = 2.39 + 0.8165Z_1 - 0.5773Z_4$$

$$M_{2(4)} = R_2 - 0.7071R_4 = 2.39 + 0.8165Z_2 - 0.5773Z_4$$

$$M_{3(4)} = R_3 + R_4 - 1.4142L = 0.49 + 0.4243Z_3 + 0.4243Z_4 - 0.8Z_7$$

$$M_{5(4)} = R_5 + 0.7071R_4 - L = 2.06 + 0.3906Z_4 + 0.5523Z_5 - 0.7365Z_7$$

$$M_{6(4)} = R_6 + 0.7071R_4 - L = 2.06 + 0.3906Z_4 + 0.5523Z_6 - 0.7365Z_7$$

$$M_5 = R_5 - 0.5L = 2.77 + 0.8321Z_5 - 0.5547Z_7$$

$$M_{1(5)} = R_1 + R_5 - L = 3.43 + 0.5145Z_1 + 0.5145Z_5 - 0.686Z_7$$

$$M_{2(5)} = R_2 + R_5 - L = 3.43 + 0.5145Z_2 + 0.5145Z_5 - 0.686Z_7$$

$$M_{3(5)} = R_3 - 1.4142R_5 = -2.39 + 0.5774Z_3 - 0.8165Z_5$$

$$M_{4(5)} = R_4 + 1.414R_5 - 1.414L = 2.06 + 0.3906Z_4 + 0.5523Z_5 - 0.7365Z_7$$

$$M_{6(5)} = R_6 - R_5 = 0 - 0.7071Z_5 + 0.7071Z_6$$

$$M_6 = R_6 - 0.5L = 2.77 + 0.8321Z_6 - 0.5547Z_7$$

$$M_{1(6)} = R_1 + R_6 - L = 3.43 + 0.5145Z_1 + 0.5145Z_6 - 0.686Z_7$$

$$M_{2(6)} = R_2 + R_6 - L = 3.43 + 0.5145Z_2 + 0.5145Z_6 - 0.686Z_7$$

$$M_{3(6)} = R_3 - 1.4142R_6 = -2.39 + 0.5774Z_3 - 0.8165Z_6$$

$$M_{4(6)} = R_4 + 1.414R_6 - 1.414L = 2.06 + 0.3906Z_4 + 0.5523Z_6 - 0.7365Z_7$$

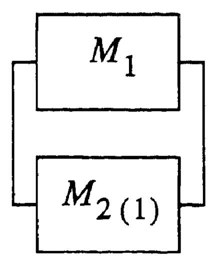
$$M_{5(6)} = R_5 - R_6 = 0 + 0.7071Z_5 - 0.7071Z_6$$

There are thirty different failure modes, and the corresponding probabilities of failure are calculated as follows:

$$M_1 = 2.77 + 0.8321Z_1 - 0.5547Z_7$$

$$M_{2(1)} = 0 - 0.7071Z_1 + 0.7071Z_7$$

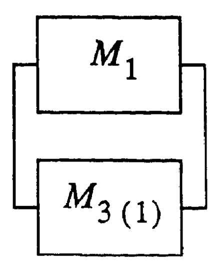
$$\rho = -0.5884, \Phi_2(-2.77, 0; -0.5884) = 0.0000393003$$



$$M_1 = 2.77 + 0.8321Z_1 - 0.5547Z_7$$

$$M_{3(1)} = 2.06 + 0.5523Z_1 + 0.3906Z_3 - 0.7365Z_7$$

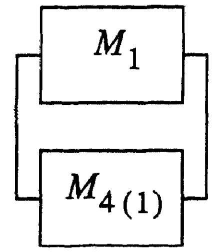
$$\rho = 0.8681, \Phi_2(-2.77, -2.06; 0.8681) = 0.00242703$$



$$M_1 = 2.77 + 0.8321Z_1 - 0.5547Z_7$$

$$M_{4\,(1)} = -2.39 - 0.8165Z_1 + 0.5774Z_4$$

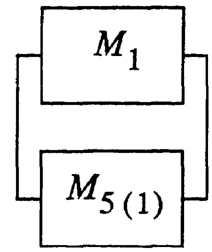
$$\rho = -0.6794,\ \Phi_2(-2.77, 2.39; -0.6794) = 0.00184232$$



$$M_1 = 2.77 + 0.8321Z_1 - 0.5547Z_7$$

$$M_{5\,(1)} = 3.43 + 0.5145Z_1 + 0.5145Z_5 - 0.686Z_7$$

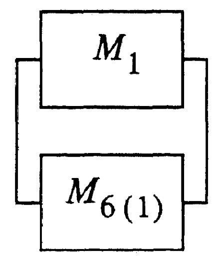
$$\rho = 0.8086,\ \Phi_2(-2.77, -3.43; 0.8086) = 0.00019021$$



$$M_1 = 2.77 + 0.8321Z_1 - 0.5547Z_7$$

$$M_{6\,(1)} = 3.43 + 0.5145Z_1 + 0.5145Z_6 - 0.686Z_7$$

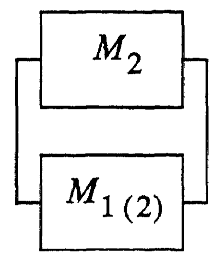
$$\rho = 0.8086,\ \Phi_2(-2.77, -3.43; 0.8086) = 0.00019021$$



$$M_2 = 2.77 + 0.8321Z_2 - 0.5547Z_7$$

$$M_{1\,(2)} = 0 + 0.7071Z_1 - 0.7071Z_2$$

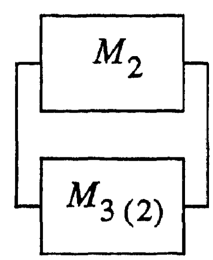
$$\rho = -0.5884,\ \Phi_2(-2.77, 0; -0.5884) = 0.0000393003$$



$$M_2 = 2.77 + 0.8321Z_2 - 0.5547Z_7$$

$$M_{3\,(2)} = 2.06 + 0.5523Z_2 + 0.3906Z_3 - 0.7365Z_7$$

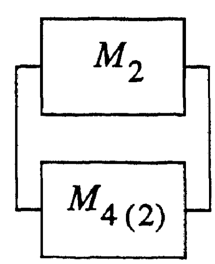
$$\rho = 0.8681,\ \Phi_2(-2.77, -2.06; 0.8681) = 0.00242703$$



$$M_2 = 2.77 + 0.8321Z_2 - 0.5547Z_7$$

$$M_{4\,(2)} = -2.39 - 0.8165Z_2 + 0.5774Z_4$$

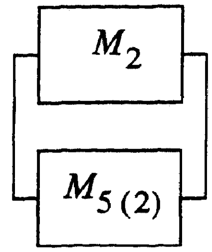
$$\rho = -0.6794,\ \Phi_2(-2.77, 2.39; -0.6794) = 0.00184232$$



$$M_2 = 2.77 + 0.8321 Z_2 - 0.5547 Z_7$$

$$M_{5\,(2)} = 3.43 + 0.5145 Z_2 + 0.5145 Z_5 - 0.686 Z_7$$

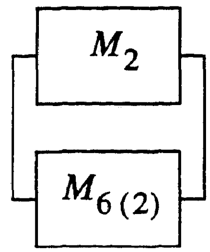
$$\rho = 0.8086,\ \Phi_2(-2.77, -3.43; 0.8086) = 0.00019021$$



$$M_2 = 2.77 + 0.8321 Z_2 - 0.5547 Z_7$$

$$M_{6\,(2)} = 3.43 + 0.5145 Z_2 + 0.5145 Z_6 - 0.686 Z_7$$

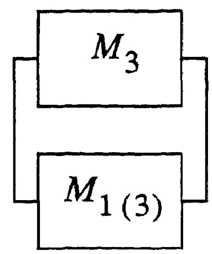
$$\rho = 0.8086,\ \Phi_2(-2.77, -3.43; 0.8086) = 0.00019021$$



$$M_3 = 0.42 + 0.7276 Z_3 - 0.686 Z_7$$

$$M_{1\,(3)} = 2.06 + 0.5523 Z_1 + 0.3906 Z_3 - 0.7365 Z_7$$

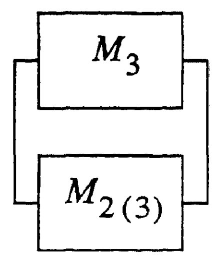
$$\rho = 0.7894,\ \Phi_2(-0.42, -2.06; 0.7894) = 0.0194841$$



$$M_3 = 0.42 + 0.7276 Z_3 - 0.686 Z_7$$

$$M_{2\,(3)} = 2.06 + 0.5523 Z_2 + 0.3906 Z_3 - 0.7365 Z_7$$

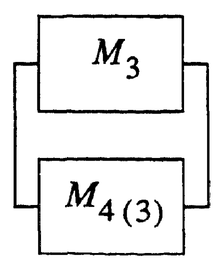
$$\rho = 0.7894,\ \Phi_2(-0.42, -2.06; 0.7894) = 0.0194841$$



$$M_3 = 0.42 + 0.7276 Z_3 - 0.686 Z_7$$

$$M_{4\,(3)} = 0.49 + 0.4243 Z_3 + 0.4243 Z_4 - 0.8 Z_7$$

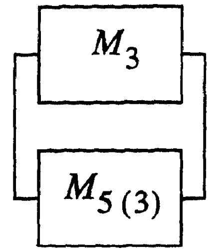
$$\rho = 0.8575,\ \Phi_2(-0.42, -0.49; 0.8575) = 0.246666$$



$$M_3 = 0.42 + 0.7276Z_3 - 0.686Z_7$$

$$M_{5\,(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_5$$

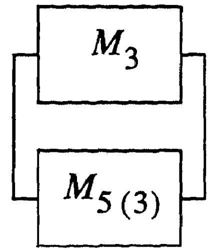
$$\rho = -0.42,\ \Phi_2(-0.42, -2.39; -0.42) = 0.000369715$$



$$M_3 = 0.42 + 0.7276Z_3 - 0.686Z_7$$

$$M_{6\,(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_6$$

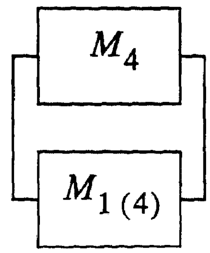
$$\rho = -0.42,\ \Phi_2(-0.42, -2.39; -0.42) = 0.000369715$$



$$M_4 = 0.42 + 0.7276Z_4 - 0.686Z_7$$

$$M_{1\,(4)} = 2.39 + 0.8165Z_1 - 0.5773Z_4$$

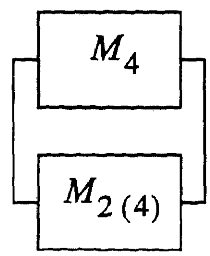
$$\rho = -0.42,\ \Phi_2(-0.42, -2.39; -0.42) = 0.000369715$$



$$M_4 = 0.42 + 0.7276Z_4 - 0.686Z_7$$

$$M_{2\,(4)} = 2.39 + 0.8165Z_2 - 0.5773Z_4$$

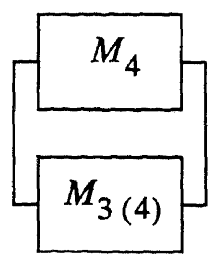
$$\rho = -0.42,\ \Phi_2(-0.42, -2.39; -0.42) = 0.000369715$$



$$M_4 = 0.42 + 0.7276Z_4 - 0.686Z_7$$

$$M_{3\,(4)} = 0.49 + 0.4243Z_3 + 0.4243Z_4 - 0.8Z_7$$

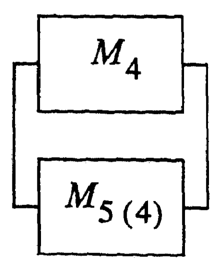
$$\rho = 0.8575,\ \Phi_2(-0.42, -0.49; 0.8575) = 0.246666$$



$$M_4 = 0.42 + 0.7276Z_4 - 0.686Z_7$$

$$M_{5\,(4)} = 2.06 + 0.3906Z_4 + 0.5523Z_5 - 0.7365Z_7$$

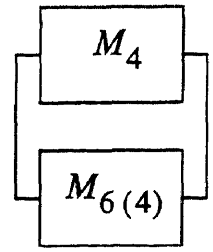
$$\rho = 0.7894,\ \Phi_2(-0.42, -2.06; 0.7894) = 0.0194841$$



$$M_4 = 0.42 + 0.7276Z_4 - 0.686Z_7$$

$$M_{6\,(4)} = 2.06 + 0.3906Z_4 + 0.5523Z_6 - 0.7365Z_7$$

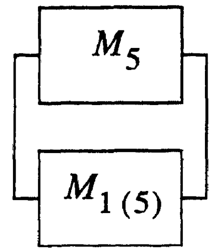
$$\rho = 0.7894, \Phi_2\,(-0.42, -2.06; 0.7894) = 0.0194841$$



$$M_5 = 2.77 + 0.8321Z_5 - 0.5547Z_7$$

$$M_{1\,(5)} = 3.43 + 0.5145Z_1 + 0.5145Z_5 - 0.686Z_7$$

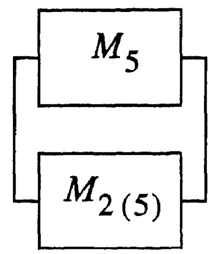
$$\rho = 0.8086, \Phi_2\,(-2.77, -3.43; 0.8086) = 0.00019021$$



$$M_5 = 2.77 + 0.8321Z_5 - 0.5547Z_7$$

$$M_{2\,(5)} = 3.43 + 0.5145Z_2 + 0.5145Z_5 - 0.686Z_7$$

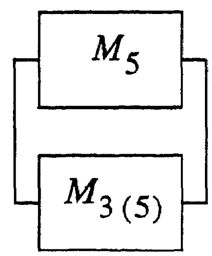
$$\rho = 0.8086, \Phi_2\,(-2.77, -3.43; 0.8086) = 0.00019021$$



$$M_5 = 2.77 + 0.8321Z_5 - 0.5547Z_7$$

$$M_{3\,(5)} = -2.39 + 0.5774Z_3 - 0.8165Z_5$$

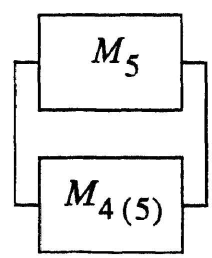
$$\rho = -0.6794, \Phi_2\,(-2.77, 2.39; -0.6974) = 0.00184232$$



$$M_5 = 2.77 + 0.8321Z_5 - 0.5547Z_7$$

$$M_{4\,(5)} = 2.06 + 0.3906Z_4 + 0.5523Z_5 - 0.7365Z_7$$

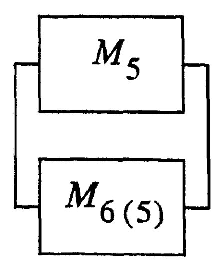
$$\rho = 0.8681, \Phi_2\,(-2.77, -2.06; 0.8681) = 0.00242703$$



$$M_5 = 2.77 + 0.8321Z_5 - 0.5547Z_7$$

$$M_{6\,(5)} = 0 - 0.7071Z_5 + 0.7071Z_6$$

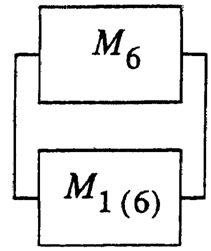
$$\rho = -0.5884, \Phi_2\,(-2.77, 0; -0.5884) = 0.0000393003$$



$$M_6 = 2.77 + 0.8321Z_6 - 0.5547Z_7$$

$$M_{1\,(6)} = 3.43 + 0.5145Z_1 + 0.5145Z_6 - 0.686Z_7$$

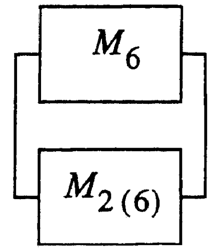
$$\rho = 0.8086,\ \Phi_2\,(-2.77, -3.43; 0.8086) = 0.00019021$$



$$M_6 = 2.77 + 0.8321Z_6 - 0.5547Z_7$$

$$M_{2\,(6)} = 3.43 + 0.5145Z_2 + 0.5145Z_6 - 0.686Z_7$$

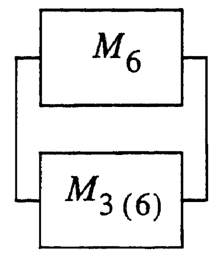
$$\rho = 0.8086,\ \Phi_2\,(-2.77, -3.43; 0.8086) = 0.00019021$$



$$M_6 = 2.77 + 0.8321Z_6 - 0.5547Z_7$$

$$M_{3\,(6)} = -2.39 + 0.5774Z_3 - 0.8165Z_6$$

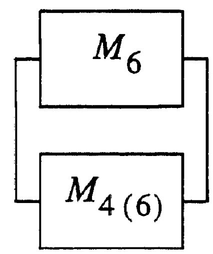
$$\rho = -0.6794,\ \Phi_2\,(-2.77, 2.39; -0.6974) = 0.00184232$$



$$M_6 = 2.77 + 0.8321Z_6 - 0.5547Z_7$$

$$M_{4\,(6)} = 2.06 + 0.3906Z_4 + 0.5523Z_6 - 0.7365Z_7$$

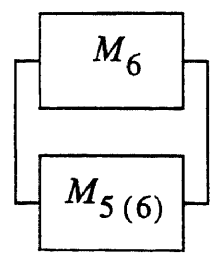
$$\rho = 0.8681,\ \Phi_2\,(-2.77, -2.06; 0.8681) = 0.00242703$$



$$M_6 = 2.77 + 0.8321Z_6 - 0.5547Z_7$$

$$M_{5\,(6)} = 0 + 0.7071Z_5 - 0.7071Z_6$$

$$\rho = -0.5884,\ \Phi_2\,(-2.77, 0; -0.5884) = 0.0000393003$$

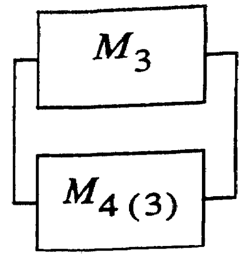


To reduce the calculation efforts, the reliability assessment is performed for the failure modes with higher probability of failure.

$$M_3 = 0.42 + 0.7276Z_3 - 0.686Z_7$$

$$M_{4(3)} = 0.49 + 0.4243Z_3 + 0.4243Z_4 - 0.8Z_7$$

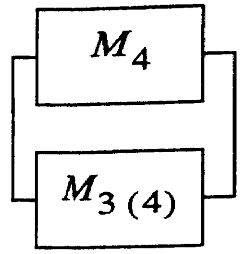
$$\rho = 0.8575, \Phi_2(-0.42, -0.49; 0.8575) = 0.246666$$



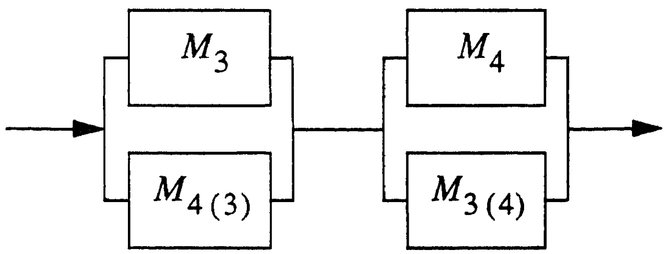
$$M_4 = 0.42 + 0.7276Z_4 - 0.686Z_7$$

$$M_{3(4)} = 0.49 + 0.4243Z_3 + 0.4243Z_4 - 0.8Z_7$$

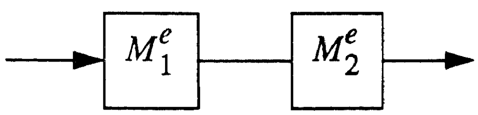
$$\rho = 0.8575, \Phi_2(-0.42, -0.49; 0.8575) = 0.246666$$



The reliability is then modelled as a series system.



and the equivalent series system:

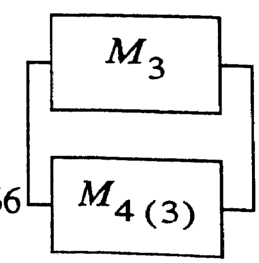


The equivalent safety margins are calculated as follows:

$$M_3 = 0.42 + 0.7276Z_3 - 0.686Z_7$$

$$M_{4(3)} = 0.49 + 0.4243Z_3 + 0.4243Z_4 - 0.8Z_7$$

$$\rho = 0.8575, P_f = \Phi_2(-0.42, -0.49; 0.8575) = 0.246666$$



$$\beta_f = 0.68$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -0.42 \\ -0.49 \end{bmatrix} - \begin{bmatrix} 0.7276 & 0 & -0.686 \\ 0.4243 & 0.4243 & -0.8 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.49 \\ -0.53 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-0.49, -0.53; 0.8575)) = -\Phi^{-1}(0.229583) = 0.74$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -0.42 \\ -0.49 \end{bmatrix} - \begin{bmatrix} 0.7276 & 0 & -0.686 \\ 0.4243 & 0.4243 & -0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.42 \\ -0.53 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-0.42, -0.53; 0.8575)) = -\Phi^{-1}(0.239473) = 0.71$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -0.42 \\ -0.49 \end{bmatrix} - \begin{bmatrix} 0.7276 & 0 & -0.686 \\ 0.4243 & 0.4243 & -0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -0.41 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-0.35, -0.41; 0.8575)) = -\Phi^{-1}(0.27168) = 0.61$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_1}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{0.74 - 0.68}{0.1} = 0.6$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_2}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{0.71-0.68}{0.1} = 0.3$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_3}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{0.61-0.68}{0.1} = -0.7$$

$$\begin{aligned}\bar{u}^e &= (u_1^e, u_2^e, u_3^e) \\ &= \left(\frac{0.6}{\sqrt{0.6^2+0.3^2+(-0.7)^2}}, \frac{0.3}{\sqrt{0.6^2+0.3^2+(-0.7)^2}}, \frac{-0.7}{\sqrt{0.6^2+0.3^2+(-0.7)^2}}\right) \\ &= (0.6189, 0.3094, -0.722)\end{aligned}$$

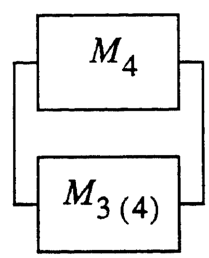
Therefore,

$$M_1^e = 0.68 + 0.6189 Z_3 + 0.3094 Z_4 - 0.722 Z_7$$

$$M_4 = 0.42 + 0.7276 Z_4 - 0.686 Z_7$$

$$M_{3(4)} = 0.49 + 0.4243 Z_3 + 0.4243 Z_4 - 0.8 Z_7$$

$$\rho = 0.8575,\ \Phi_2(-0.42, -0.49; 0.8575) = 0.246666$$



$$\beta_f = 0.68$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix}-0.42 \\ -0.49\end{bmatrix} - \begin{bmatrix}0 & 0.7276 & -0.686 \\ 0.4243 & 0.4243 & -0.8\end{bmatrix}\begin{bmatrix}0.1 \\ 0 \\ 0\end{bmatrix} = \begin{bmatrix}-0.42 \\ -0.53\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-0.42, -0.53; 0.8575)) = -\Phi^{-1}(0.23948) = 0.71$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -0.42 \\ -0.49 \end{bmatrix} - \begin{bmatrix} 0 & 0.7276 & -0.686 \\ 0.4243 & 0.4243 & -0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.49 \\ -0.53 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-0.49, -0.53; 0.8575)) = -\Phi^{-1}(0.22961) = 0.74$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -0.42 \\ -0.49 \end{bmatrix} - \begin{bmatrix} 0 & 0.7276 & -0.686 \\ 0.4243 & 0.4243 & -0.8 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -0.35 \\ -0.41 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-0.35, -0.41; 0.8575)) = -\Phi^{-1}(0.27169) = 0.61$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_1}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{0.71 - 0.68}{0.1} = 0.3$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_2}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{0.74 - 0.68}{0.1} = 0.6$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_3}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{0.61 - 0.68}{0.1} = -0.7$$

$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

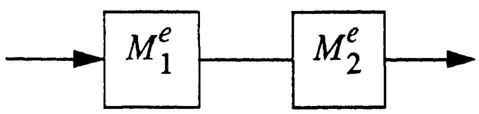
$$= \left(\frac{0.3}{\sqrt{0.6^2+0.3^2+(-0.7)^2}}, \frac{0.6}{\sqrt{0.6^2+0.3^2+(-0.7)^2}}, \frac{-0.7}{\sqrt{0.6^2+0.3^2+(-0.7)^2}}\right)$$

$$= (0.3094, 0.6189, -0.722)$$

Therefore,

$$M_2^e = 0.68 + 0.3094Z_3 + 0.6189Z_4 - 0.722Z_7$$

The Ditlevsen bounds for the probability of failure of the series system are as follows:



$$M_1^e = 0.68 + 0.6189Z_3 + 0.3094Z_4 - 0.722Z_7$$

$$M_2^e = 0.68 + 0.3094Z_3 + 0.6189Z_4 - 0.722Z_7, \rho_{12} = 0.9043$$

upper bound:

$$P_f \le \Phi(-\beta_1) + \Phi(-\beta_2) - \Phi_2(-\beta_2, -\beta_1; \rho)$$

$$= \Phi(-0.68) + \Phi(-0.68) - \Phi_2(-0.68, -0.68; 0.9043)$$

$$= 0.24667 + 0.24667 - 0.192757 = 0.3006$$

The lower bound:

$$P_f \geq \Phi(-\beta_1) + max\,[\Phi(-\beta_2) - \Phi_2(-\beta_2, -\beta_1; \rho), 0]$$

$$= \Phi(-0.68) + \Phi(-0.68) - \Phi_2(-0.68, -0.68; 0.9043)$$

$$= 0.24667 + 0.24667 - 0.192757 = 0.3006$$

Therefore, an estimate of the structural reliability is

$$P_f = 0.3006,\ \beta = -\Phi^{-1}(0.3006) = 0.52$$

Example 3 Consider the same structure of example 2 with different external load $\mu_L = 60000, \sigma_L = 6000$. Calculate the structural reliability.

By the same procedure used in example 2, one can obtain the following results:

For L=60000, the ANSYS PROGRAMS and the corresponding results are listed in the appendixes: E,E1,E2,E3,E4,E5,J,J1,J2,J3,J4,J5,J6,K,K1,K2,K3,K4,and K5.

$$M_3 = R_3^{\ +} \ - S_3\,(tesnion) = R_3 - 42426 = R_3 - 0.7071L$$

$$M_{5\,(3)} = R_5^- \ + S_5\,(compression) = R_5 - 42426 = R_5 - 0.7071R_3$$

$$M_{6\,(3)} = R_6^- \ + S_6\,(compression) = R_6 - 42426 = R_6 - 0.7071R_3$$

$$M_4 = R_4^- \ + S_4\,(compression) = R_4 - 42426 = R_4 - 0.7071L$$

$$M_{1\,(4)} = R_1^{\ +} \ - S_1\,(tesnion) = R_1 - 42426 = R_1 - 0.7071R_4$$

$$M_{2(4)} = R_2^{+} - S_2(tesnion) = R_2 - 42426 = R_2 - 0.7071R_4$$

For the normalization: $Z_i = \dfrac{X_i - \mu_{X_i}}{\sigma_{X_i}}, X_i = \sigma_{X_i} Z_i + \mu_{X_i}$

$$R_i = \sigma_{R_i} Z_i + \mu_{R_i} = 6000Z_i + 60000, \ (i = 1, \ldots, 6).$$

$$L = \sigma_L Z_7 + \mu_L = 6000Z_7 + 60000$$

$$M_3 = R_3 - 0.7071L = (6000Z_3 + 60000) - 0.7071\,(6000Z_7 + 60000)$$

$$\Rightarrow 2.39 + 0.8165Z_3 - 0.5773Z_7$$

$$M_{5(3)} = R_5 - 0.7071R_3 = (6000Z_5 + 60000) - 0.7071\,(6000Z_3 + 60000)$$

$$\Rightarrow 2.39 - 0.5773Z_3 + 0.8165Z_5$$

$$M_{6(3)} = R_6 - 0.7071R_3 = (6000Z_6 + 60000) - 0.7071\,(6000Z_3 + 60000)$$

$$\Rightarrow 2.39 - 0.5773Z_3 + 0.8165Z_6$$

$$M_4 = R_4 - 0.7071L = (6000Z_4 + 60000) - 0.7071\,(6000Z_7 + 60000)$$

$$\Rightarrow 2.39 + 0.8165Z_4 - 0.5773Z_7$$

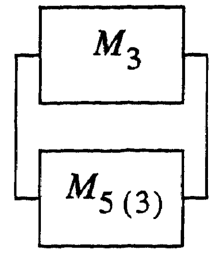
$$M_{1(4)} = R_1 - 0.7071R_4 = (6000Z_1 + 60000) - 0.7071\,(6000Z_4 + 60000)$$

$$\Rightarrow 2.39 + 0.8165Z_1 - 0.5773Z_4$$

$$M_{2(4)} = R_2 - 0.7071R_4 = (6000Z_2 + 60000) - 0.7071\,(6000Z_4 + 60000)$$

$$\Rightarrow 2.39 + 0.8165Z_2 - 0.5773Z_4$$

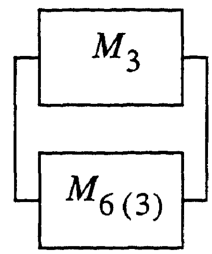
$$M_3 = 2.39 + 0.8165Z_3 - 0.5773Z_7$$



$$M_{5(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_5,\ \rho = -0.4714$$

$$\Phi_2(-2.39, -2.39; -0.4714) = 0.1566\times10^{-6},\ \beta = 5.12$$

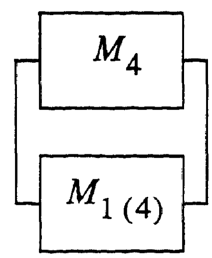
$$M_3 = 2.39 + 0.8165Z_3 - 0.5773Z_7$$



$$M_{6(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_6,\ \rho = -0.4714$$

$$\Phi_2(-2.39, -2.39; -0.4714) = 0.1566\times10^{-6},\ \beta = 5.12$$

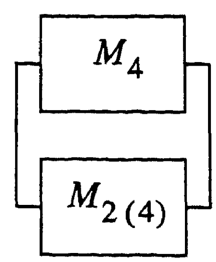
$$M_4 = 2.39 + 0.8165Z_4 - 0.5773Z_7$$



$$M_{1(4)} = 2.39 + 0.8165Z_4 - 0.5773Z_7,\ \rho = -0.4714$$

$$\Phi_2(-2.39, -2.39; -0.4714) = 0.1566\times10^{-6},\ \beta = 5.12$$

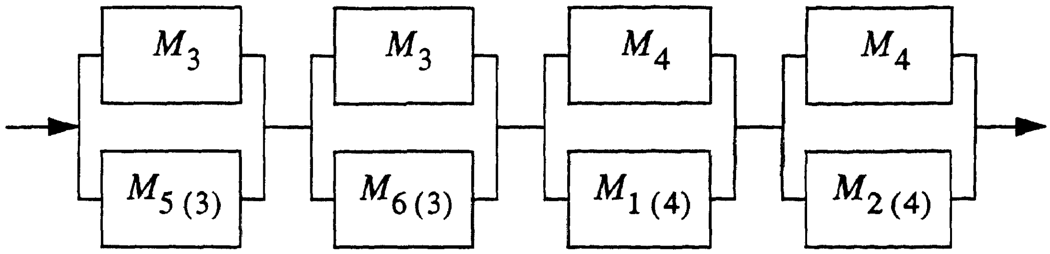
$$M_4 = 2.39 + 0.8165Z_4 - 0.5773Z_7$$



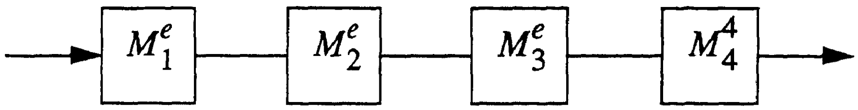
$$M_{2(4)} = 2.39 + 0.8165Z_2 - 0.5773Z_4,\ \rho = -0.4714$$

$$\Phi_2(-2.39, -2.39; -0.4714) = 0.1566\times10^{-6},\ \beta = 5.12$$

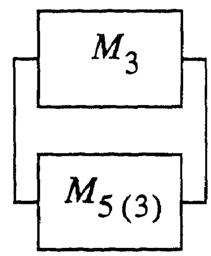
The reliability is modelled as a series system:



The equivalent series system is



The equivalent safety margins are calculatedd as follows:



$$M_3 = 2.39 + 0.8165 Z_3 - 0.5773 Z_7$$

$$M_{5\,(3)} = 2.39 - 0.5773 Z_3 + 0.8165 Z_5,\ \rho = -0.4714$$

$$\Phi_2(-2.39, -2.39; -0.4714) = 0.1566\times 10^{-6},\ \beta = 5.12$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -2.39 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.8165 & 0 & -0.5773 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2.47 \\ -2.33 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.47, -2.33; -0.47)) = -\Phi^{-1}(0.1415\times 10^{-6}) = 5.14$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -2.39 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.8165 & 0 & -0.5773 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2.39 \\ -2.47 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.39, -2.47; -0.47)) = -\Phi^{-1}(0.1055 \times 10^{-6}) = 5.19$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -2.39 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.8165 & 0 & -0.5773 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -2.33 \\ -2.39 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.33, -2.39; -0.47)) = -\Phi^{-1}(0.2095 \times 10^{-6}) = 5.06$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_1}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{5.14 - 5.12}{0.1} = 0.2$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_2}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{5.19 - 5.12}{0.1} = 0.7$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_3}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{5.06 - 5.12}{0.1} = -0.6$$

$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

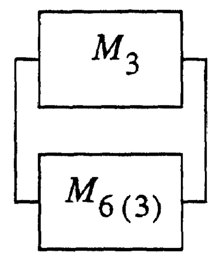
$$= \left(\frac{0.2}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}, \frac{0.7}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}, \frac{-0.6}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}\right)$$

$$= (0.212, 0.742, -0.636)$$

Therefore,

$$M_1^e = 5.12 + 0.212Z_3 + 0.741Z_5 - 0.636Z_7$$

$$M_3 = 2.39 + 0.8165Z_3 - 0.5773Z_7$$



$$M_{6(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_6, \rho = -0.4714$$

$$\Phi_2(-2.39, -2.39; -0.4714) = 0.1566\times10^{-6}, \beta = 5.12$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix}-2.39\\-2.39\end{bmatrix} - \begin{bmatrix}0.8165 & 0 & -0.5773\\-0.5773 & 0.8165 & 0\end{bmatrix}\begin{bmatrix}0.1\\0\\0\end{bmatrix} = \begin{bmatrix}-2.47\\-2.33\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.47, -2.33; -0.47)) = -\Phi^{-1}(0.1415\times10^{-6}) = 5.14$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-2.39\\-2.39\end{bmatrix} - \begin{bmatrix}0.8165 & 0 & -0.5773\\-0.5773 & 0.8165 & 0\end{bmatrix}\begin{bmatrix}0\\0.1\\0\end{bmatrix} = \begin{bmatrix}-2.39\\-2.47\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.39, -2.47; -0.47)) = -\Phi^{-1}(0.1055\times10^{-6}) = 5.19$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-2.39\\-2.39\end{bmatrix} - \begin{bmatrix}0.8165 & 0 & -0.5773\\-0.5773 & 0.8165 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.1\end{bmatrix} = \begin{bmatrix}-2.33\\-2.39\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.33, -2.39; -0.47)) = -\Phi^{-1}(0.2095\times10^{-6}) = 5.06$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_1}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{5.14-5.12}{0.1} = 0.2$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_2}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{5.19-5.12}{0.1} = 0.7$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_3}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{5.06-5.12}{0.1} = -0.6$$

$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

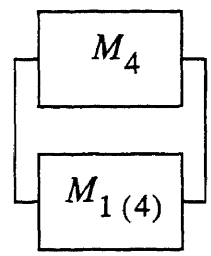
$$= \left(\frac{0.2}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}, \frac{0.7}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}, \frac{-0.6}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}\right)$$

$$= (0.212, 0.742, -0.636)$$

Therefore,

$$M_2^e = 5.12 + 0.212Z_3 + 0.741Z_6 - 0.636Z_7$$

$$M_4 = 2.39 + 0.8165Z_4 - 0.5773Z_7$$



$$M_{1\,(4)} = 2.39 + 0.8165Z_1 - 0.5773Z_4,\ \rho = -0.4714$$

$$\Phi_2(-2.39, -2.39; -0.4714) = 0.1566\times 10^{-6},\ \beta = 5.12$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -2.39 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0 & 0.8165 & -0.5773 \\ 0.8165 & -0.5773 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2.39 \\ -2.47 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.39, -2.47; -0.47)) = -\Phi^{-1}(0.1055\times 10^{-6}) = 5.19$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-2.39\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.8165 & -0.5773\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0\\0.1\\0\end{bmatrix} = \begin{bmatrix}-2.47\\-2.33\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.47,-2.33;-0.47)) = -\Phi^{-1}(0.1415\times10^{-6}) = 5.14$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-2.39\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.8165 & -0.5773\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.1\end{bmatrix} = \begin{bmatrix}-2.33\\-2.39\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.33,-2.39;-0.47)) = -\Phi^{-1}(0.2095\times10^{-6}) = 5.06$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_1}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{5.19-5.12}{0.1} = 0.7$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_2}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{5.14-5.12}{0.1} = 0.2$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_3}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{5.06-5.12}{0.1} = -0.6$$

$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

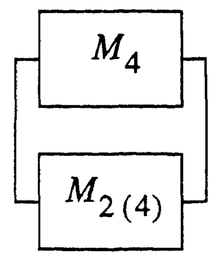
$$= \left(\frac{0.7}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}, \frac{0.2}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}, \frac{-0.6}{\sqrt{0.2^2+0.7^2+(-0.6)^2}}\right)$$

$$= (0.742, 0.212, -0.636)$$

Therefore,

$$M_3^e = 5.12 + 0.742Z_1 + 0.212Z_4 - 0.636Z_7$$

$$M_4 = 2.39 + 0.8165Z_4 - 0.5773Z_7$$



$$M_{2(4)} = 2.39 + 0.8165Z_2 - 0.5773Z_4,\ \rho = -0.4714$$

$$\Phi_2(-2.39, -2.39; -0.4714) = 0.1566\times10^{-6},\ \beta = 5.12$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix}-2.39\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.8165 & -0.5773\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0.1\\0\\0\end{bmatrix} = \begin{bmatrix}-2.39\\-2.47\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.39, -2.47; -0.47)) = -\Phi^{-1}(0.1055\times10^{-6}) = 5.19$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -2.39 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0 & 0.8165 & -0.5773 \\ 0.8165 & -0.5773 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2.47 \\ -2.33 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.47, -2.33; -0.47)) = -\Phi^{-1}(0.1415 \times 10^{-6}) = 5.14$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -2.39 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0 & 0.8165 & -0.5773 \\ 0.8165 & -0.5773 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -2.33 \\ -2.39 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-2.33, -2.39; -0.47)) = -\Phi^{-1}(0.2095 \times 10^{-6}) = 5.06$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_1}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{5.19 - 5.12}{0.1} = 0.7$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_2}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{5.14 - 5.12}{0.1} = 0.2$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_3}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{5.06 - 5.12}{0.1} = -0.6$$

$$\vec{u}^e = (u_1^e, u_2^e, u_3^e)$$

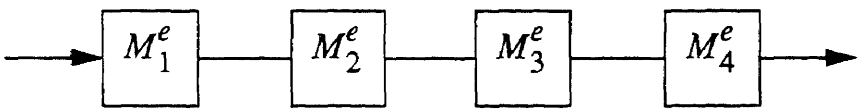
$$= \left(\frac{0.7}{\sqrt{0.2^2 + 0.7^2 + (-0.6)^2}}, \frac{0.2}{\sqrt{0.2^2 + 0.7^2 + (-0.6)^2}}, \frac{-0.6}{\sqrt{0.2^2 + 0.7^2 + (-0.6)^2}}\right)$$

$$= (0.742, 0.212, -0.636)$$

Therefore,

$$M_4^e = 5.12 + 0.742Z_2 + 0.212Z_4 - 0.636Z_7$$

The Ditlevsen bounds for the probability of failure of the series system are:



$$M_1^e = 5.12 + 0.212Z_3 + 0.741Z_5 - 0.636Z_7, \rho_{12} = 0.4494, \rho_{13} = 0.4045$$

$$M_2^e = 5.12 + 0.212Z_3 + 0.741Z_6 - 0.636Z_7, \rho_{14} = 0.4045, \rho_{23} = 0.4045$$

$$M_3^e = 5.12 + 0.742Z_1 + 0.212Z_4 - 0.636Z_7, \rho_{24} = 0.4045, \rho_{34} = 0.4494$$

$$M_4^e = 5.12 + 0.742Z_2 + 0.212Z_4 - 0.636Z_7$$

upper bound: $P_f \le \sum_{i=1}^{n} \Phi(-\beta_i) - \sum_{i=2, j<i}^{n} max\Phi_2(-\beta_i, -\beta_j; \rho)$

$$P_f \le \Phi(-\beta_1) + \Phi(-\beta_2) + \Phi(-\beta_3) + \Phi(-\beta_4) - max[\Phi_2(-\beta_2, -\beta_1; \rho)]$$

$$-max[\Phi_2(-\beta_3, -\beta_2; \rho), \Phi_2(-\beta_3, -\beta_1; \rho)]$$

$$-max[\Phi(-\beta_4, -\beta_1; \rho), \Phi(-\beta_4, -\beta_2; \rho), \Phi(-\beta_4, -\beta_3; \rho)]$$

$$= 4 \times 0.1536 \times 10^{-6} - 1.7501 \times 10^{-10} - 9.0809 \times 10^{-11} - 1.7501 \times 10^{-10}$$

$$= 6.1396 \times 10^{-7}$$

lower bound: $P_f \ge \Phi(-\beta_1) + \sum_{i=2}^{n} max \left[\Phi(-\beta_i) - \sum_{j=1}^{i-1} \Phi_2(-\beta_i, -\beta_j; \rho), 0 \right]$

$$P_f \ge \Phi(-\beta_1) + max[\Phi(-\beta_2) - \Phi_2(-\beta_2, -\beta_1; \rho), 0]$$

$$+ max[\Phi(-\beta_3) - \Phi_2(-\beta_3, -\beta_1; \rho) - \Phi_2(-\beta_3, -\beta_2; \rho), 0] +$$

$$max[\Phi(-\beta_4) - \Phi_2(-\beta_4, -\beta_1; \rho) - \Phi_2(-\beta_4, -\beta_2; \rho) - \Phi_2(-\beta_4, -\beta_3; \rho), 0]$$

$$= 0.1536 \times 10^{-6} + 0.1536 \times 10^{-6} - 1.7501 \times 10^{-10} + 0.1536 \times 10^{-6}$$

$$- 9.0809 \times 10^{-11} - 9.0809 \times 10^{-11} + 0.1536 \times 10^{-6} - 9.0809 \times 10^{-11}$$

$$- 9.0809 \times 10^{-11} - 1.7501 \times 10^{-10} = 6.1369 \times 10^{-7}$$

$$6.1369 \times 10^{-7} \le P_f \le 6.1396 \times 10^{-7}$$

$$P_f \approx 6.1383 \times 10^{-7}, \beta = 4.85$$

Example 4. Consider the same structure of example 2 with different external load $\mu_L = 40000, \sigma_L = 4000$. Calculate the structural reliability.

By the same procedure used in example 2, one can obtain the following results:
For L=40000, the ANSYS PROGRAMS and the corresponding results are listed in the appendixes: E,E1,E2,E3,E4,E5,H,H1,H2,H3,H4,H5,H6,I,I1,I2,I3,I4,and I5.

$$M_3 = R_3{}^+ - S_3\,(tesnion) = R_3 - 28284 = R_3 - 0.7071L$$

$$M_{5\,(3)} = R_5^- + S_5\,(compression) = R_5 - 42426 = R_5 - 0.7071R_3$$

$$M_{6\,(3)} = R_6^- + S_6\,(compression) = R_6 - 42426 = R_6 - 0.7071R_3$$

$$M_4 = R_4^- + S_4\,(compression) = R_4 - 28284 = R_4 - 0.7071L$$

$$M_{1\,(4)} = R_1{}^+ - S_1\,(tesnion) = R_1 - 42426 = R_1 - 0.7071R_4$$

$$M_{2\,(4)} = R_2{}^+ - S_2\,(tesnion) = R_2 - 42426 = R_2 - 0.7071R_4$$

For the normalization: $Z_i = \dfrac{X_i - \mu_{X_i}}{\sigma_{X_i}}, X_i = \sigma_{X_i} Z_i + \mu_{X_i}$

$$R_i = \sigma_{R_i} Z_i + \mu_{R_i} = 6000Z_i + 60000,\ (i = 1, \ldots, 6).$$

$$L = \sigma_L Z_7 + \mu_L = 4000Z_7 + 40000$$

$$M_3 = R_3 - 0.7071L = (6000Z_3 + 60000) - 0.7071\,(4000Z_7 + 40000)$$

$$\Rightarrow 4.78 + 0.9045Z_3 - 0.4264Z_7$$

$$M_{5\,(3)} = R_5 - 0.7071R_3 = (6000Z_5 + 60000) - 0.7071\,(6000Z_3 + 60000)$$

$$\Rightarrow 2.39 - 0.5773Z_3 + 0.8165Z_5$$

$$M_{6\,(3)} = R_6 - 0.7071R_3 = (6000Z_6 + 60000) - 0.7071\,(6000Z_3 + 60000)$$

$$\Rightarrow 2.39 - 0.5773Z_3 + 0.8165Z_6$$

$$M_4 = R_4 - 0.7071L = (6000Z_4 + 60000) - 0.7071\,(4000Z_7 + 40000)$$

$$\Rightarrow 4.78 + 0.9045Z_4 - 0.4264Z_7$$

$$M_{1\,(4)} = R_1 - 0.7071R_4 = (6000Z_1 + 60000) - 0.7071\,(6000Z_4 + 60000)$$

$$\Rightarrow 2.39 + 0.8165Z_1 - 0.5773Z_4$$

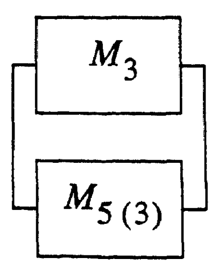
$$M_{2\,(4)} = R_2 - 0.7071R_4 = (6000Z_2 + 60000) - 0.7071\,(6000Z_4 + 60000)$$

$$\Rightarrow 2.39 + 0.8165Z_2 - 0.5773Z_4$$

$$M_3 = 4.78 + 0.9045Z_3 - 0.4264Z_7$$

$$M_{5\,(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_5,\ \rho = -0.5222$$

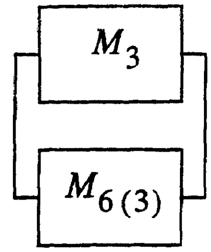
$$\Phi_2\,(-4.78, -2.39; -0.5222) = 0.2571\times10^{-14},\ \beta = 7.83$$



$$M_3 = 4.78 + 0.9045Z_3 - 0.4264Z_7$$

$$M_{6(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_6,\ \rho = -0.5222$$

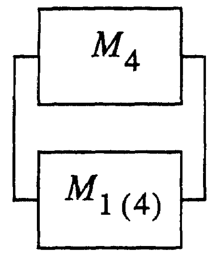
$$\Phi_2(-4.78, -2.39; -0.5222) = 0.2571\times10^{-14},\ \beta = 7.83$$



$$M_4 = 4.78 + 0.9045Z_4 - 0.4264Z_7,$$

$$M_{1(4)} = 2.39 + 0.8165Z_4 - 0.5773Z_7,\ \rho = -0.5222$$

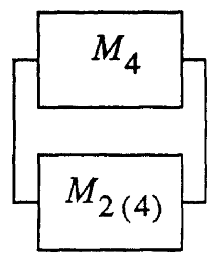
$$\Phi_2(-4.78, -2.39; -0.5222) = 0.2571\times10^{-14},\ \beta = 7.83$$



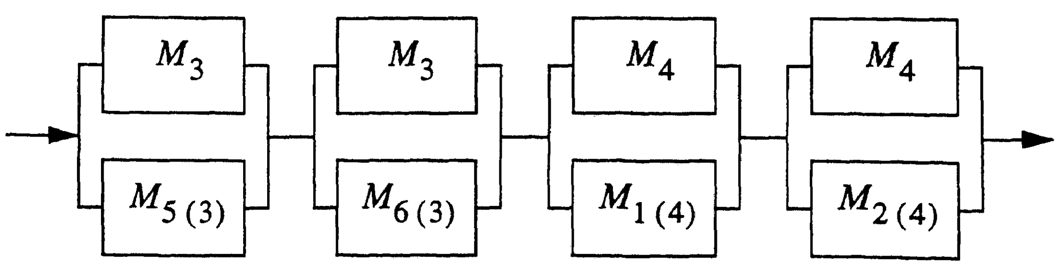
$$M_4 = 4.78 + 0.9045Z_4 - 0.4264Z_7$$

$$M_{2(4)} = 2.39 + 0.8165Z_2 - 0.5773Z_4,\ \rho = -0.5222$$

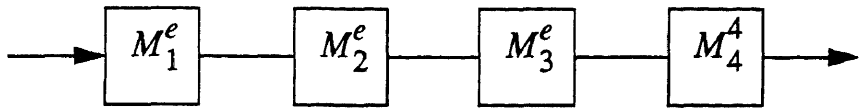
$$\Phi_2(-4.78, -2.39; -0.5222) = 0.2571\times10^{-14},\ \beta = 7.83$$



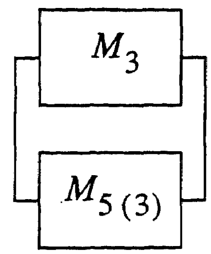
The reliability is modelled as a series system:



The equivalent series system is



The equivalent safety margins are calculated as follows:



$$M_3 = 4.78 + 0.9045 Z_3 - 0.4264 Z_7$$

$$M_{5\,(3)} = 2.39 - 0.5773 Z_3 + 0.8165 Z_5,\ \rho = -0.5222$$

$$\Phi_2(-4.78, -2.39; -0.5222) = 0.2571 \times 10^{-14},\ \beta = 7.83$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -4.78 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9045 & 0 & -0.4264 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.87 \\ -2.33 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.87, -2.33; -0.52)) = -\Phi^{-1}(0.1805 \times 10^{-14}) = 7.87$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -4.78 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9045 & 0 & -0.4264 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.78 \\ -2.47 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.78, -2.47; -0.52)) = -\Phi^{-1}(0.1464 \times 10^{-14}) = 7.89$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -4.78 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9045 & 0 & -0.4264 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -4.74 \\ -2.39 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.74, -2.39; -0.52)) = -\Phi^{-1}(0.3616 \times 10^{-14}) = 7.78$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_1}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{7.87 - 7.83}{0.1} = 0.4$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_2}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{7.89 - 7.83}{0.1} = 0.6$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_3}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{7.78 - 7.83}{0.1} = -0.5$$

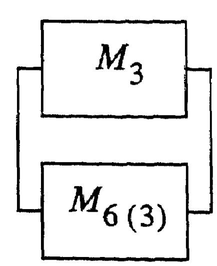
$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

$$= \left(\frac{0.4}{\sqrt{0.4^2 + 0.6^2 + (-0.5)^2}}, \frac{0.6}{\sqrt{0.4^2 + 0.6^2 + (-0.5)^2}}, \frac{-0.5}{\sqrt{0.4^2 + 0.6^2 + (-0.5)^2}} \right)$$

$$= (0.4558, 0.6838, -0.5698)$$

Therefore,

$$M_1^e = 7.83 + 0.4558 Z_3 + 0.6838 Z_5 - 0.5698 Z_7$$



$$M_3 = 4.78 + 0.9045 Z_3 - 0.4264 Z_7$$

$$M_{6(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_6, \rho = -0.5222$$

$$\Phi_2(-4.78, -2.39; -0.5222) = 0.2571\times10^{-14}, \beta = 7.83$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix}-4.78\\-2.39\end{bmatrix} - \begin{bmatrix}0.9045 & 0 & -0.4264\\-0.5773 & 0.8165 & 0\end{bmatrix}\begin{bmatrix}0.1\\0\\0\end{bmatrix} = \begin{bmatrix}-4.87\\-2.33\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.87, -2.33; -0.52)) = -\Phi^{-1}(0.1805\times10^{-14}) = 7.87$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix}-4.78\\-2.39\end{bmatrix} - \begin{bmatrix}0.9045 & 0 & -0.4264\\-0.5773 & 0.8165 & 0\end{bmatrix}\begin{bmatrix}0\\0.1\\0\end{bmatrix} = \begin{bmatrix}-4.78\\-2.47\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.78, -2.47; -0.52)) = -\Phi^{-1}(0.1464\times10^{-14}) = 7.89$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-4.78\\-2.39\end{bmatrix} - \begin{bmatrix}0.9045 & 0 & -0.4264\\-0.5773 & 0.8165 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.1\end{bmatrix} = \begin{bmatrix}-4.74\\-2.39\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.74, -2.39; -0.52)) = -\Phi^{-1}(0.3616\times 10^{-14}) = 7.78$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_1}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{7.87-7.83}{0.1} = 0.4$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_2}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{7.89-7.83}{0.1} = 0.6$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_3}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{7.78-7.83}{0.1} = -0.5$$

$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

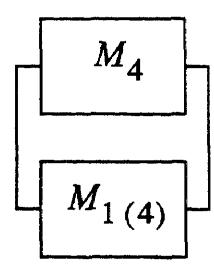
$$= \left(\frac{0.4}{\sqrt{0.4^2+0.6^2+(-0.5)^2}}, \frac{0.6}{\sqrt{0.4^2+0.6^2+(-0.5)^2}}, \frac{-0.5}{\sqrt{0.4^2+0.6^2+(-0.5)^2}}\right)$$

$$= (0.4558, 0.6838, -0.5698)$$

Therefore, $M_2^e = 7.83 + 0.4558Z_3 + 0.6838Z_5 - 0.5698Z_7$

$M_4 = 4.78 + 0.9045Z_4 - 0.4264Z_7$

$M_{1(4)} = 2.39 + 0.8165Z_1 - 0.5773Z_4$, $\rho = -0.5222$



$$\Phi_2(-4.78,-2.39;-0.5222) = 0.2571\times10^{-14},\ \beta = 7.83$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-4.78\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9045 & -0.4264\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0.1\\0\\0\end{bmatrix} = \begin{bmatrix}-4.78\\-2.47\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.78,-2.47;-0.52)) = -\Phi^{-1}(0.1464\times10^{-14}) = 7.89$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-4.78\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9045 & -0.4264\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0\\0.1\\0\end{bmatrix} = \begin{bmatrix}-4.87\\-2.33\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.87,-2.33;-0.52)) = -\Phi^{-1}(0.1805\times10^{-14}) = 7.87$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-4.78\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9045 & -0.4264\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.1\end{bmatrix} = \begin{bmatrix}-4.74\\-2.39\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.74, -2.39; -0.52)) = -\Phi^{-1}(0.3616\times10^{-14}) = 7.78$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_1}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{7.89-7.83}{0.1} = 0.6$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_2}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{7.87-7.83}{0.1} = 0.4$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_3}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{7.78-7.83}{0.1} = -0.5$$

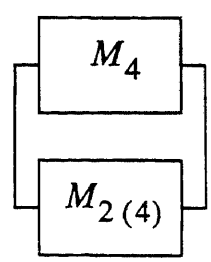
$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

$$= \left(\frac{0.6}{\sqrt{0.4^2+0.6^2+(-0.5)^2}}, \frac{0.4}{\sqrt{0.4^2+0.6^2+(-0.5)^2}}, \frac{-0.5}{\sqrt{0.4^2+0.6^2+(-0.5)^2}}\right)$$

$$= (0.6838, 0.4558, -0.5698)$$

Therefore, $M_3^e = 7.83 + 0.6838Z_1 + 0.4558Z_4 - 0.5698Z_7$

$$M_4 = 4.78 + 0.9045Z_4 - 0.4264Z_7$$



$$M_{2(4)} = 2.39 + 0.8165Z_2 - 0.5773Z_4,\ \rho = -0.5222$$

$$\Phi_2(-4.78, -2.39; -0.5222) = 0.2571\times10^{-14},\ \beta = 7.83$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -4.78 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0 & 0.9045 & -0.4264 \\ 0.8165 & -0.5773 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.78 \\ -2.47 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.78, -2.47; -0.52)) = -\Phi^{-1}(0.1464 \times 10^{-14}) = 7.89$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -4.78 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0 & 0.9045 & -0.4264 \\ 0.8165 & -0.5773 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4.87 \\ -2.33 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.87, -2.33; -0.52)) = -\Phi^{-1}(0.1805 \times 10^{-14}) = 7.87$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -4.78 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0 & 0.9045 & -0.4264 \\ 0.8165 & -0.5773 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -4.74 \\ -2.39 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-4.74, -2.39; -0.52)) = -\Phi^{-1}(0.3616 \times 10^{-14}) = 7.78$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_1}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{7.89 - 7.83}{0.1} = 0.6$$

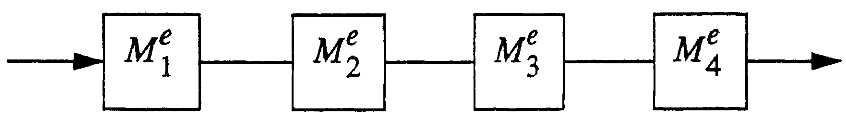
$$\left.\frac{\partial \beta_f}{\partial \varepsilon_2}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{7.87 - 7.83}{0.1} = 0.4$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_3}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{7.78 - 7.83}{0.1} = -0.5$$

$$\begin{aligned}
\bar{u}^e &= (u_1^e, u_2^e, u_3^e) \\
&= \left(\frac{0.6}{\sqrt{0.4^2 + 0.6^2 + (-0.5)^2}}, \frac{0.4}{\sqrt{0.4^2 + 0.6^2 + (-0.5)^2}}, \frac{-0.5}{\sqrt{0.4^2 + 0.6^2 + (-0.5)^2}}\right) \\
&= (0.6838, 0.4558, -0.5698)
\end{aligned}$$

Therefore, $M_4^e = 7.83 + 0.6838 Z_1 + 0.4558 Z_4 - 0.5698 Z_7$

The Ditlevsen bounds for the probability of failure of the series system are:



$M_1^e = 7.83 + 0.4558 Z_3 + 0.6838 Z_5 - 0.5698 Z_7$, $\rho_{12} = 0.5324$,

$\rho_{13} = 0.3247$

$M_2^e = 7.83 + 0.4558 Z_3 + 0.6838 Z_5 - 0.5698 Z_7$, $\rho_{14} = 0.3247$,

$\rho_{23} = 0.3247$

$M_3^e = 7.83 + 0.6838Z_1 + 0.4558Z_4 - 0.5698Z_7,$ $\rho_{24} = 0.3247,$

$\rho_{34} = 0.5324$

$$M_4^e = 7.83 + 0.6838Z_2 + 0.4558Z_4 - 0.5698Z_7$$

upper bound: $P_f \le \sum_{i=1}^{n} \Phi(-\beta_i) - \sum_{i=2, j<i}^{n} max\Phi_2(-\beta_i, -\beta_j; \rho)$

$$P_f \le \Phi(-\beta_1) + \Phi(-\beta_2) + \Phi(-\beta_3) + \Phi(-\beta_4) - max[\Phi_2(-\beta_2, -\beta_1; \rho)]$$

$$-max[\Phi_2(-\beta_3, -\beta_2; \rho), \Phi_2(-\beta_3, -\beta_1; \rho)]$$

$$-max[\Phi(-\beta_4, -\beta_1; \rho), \Phi(-\beta_4, -\beta_2; \rho), \Phi(-\beta_4, -\beta_3; \rho)]$$

$$= 4 \times 0.2439 \times 10^{-14} - 0.2826 \times 10^{-19} - 3.641 \times 10^{-23} - 0.2826 \times 10^{-19}$$

$$= 9.7559 \times 10^{-15}$$

lower bound: $P_f \ge \Phi(-\beta_1) + \sum_{i=2}^{n} max \left[\Phi(-\beta_i) - \sum_{j=1}^{i-1} \Phi_2(-\beta_i, -\beta_j; \rho), 0 \right]$

$$P_f \ge \Phi(-\beta_1) + max[\Phi(-\beta_2) - \Phi_2(-\beta_2, -\beta_1; \rho), 0]$$

$$+ max[\Phi(-\beta_3) - \Phi_2(-\beta_3, -\beta_1; \rho) - \Phi_2(-\beta_3, -\beta_2; \rho), 0] +$$

$$max[\Phi(-\beta_4) - \Phi_2(-\beta_4, -\beta_1; \rho) - \Phi_2(-\beta_4, -\beta_2; \rho) - \Phi_2(-\beta_4, -\beta_3; \rho), 0]$$

$$= 0.2439 \times 10^{-14} + 0.2439 \times 10^{-14} - 0.2826 \times 10^{-19} + 0.2439 \times 10^{-14}$$

$$- 3.641\times10^{-23} - 3.641\times10^{-23} + 0.2439\times10^{-14} - 3.641\times10^{-23}$$

$$- 3.641\times10^{-23} - 0.2826\times10^{-19} = 9.7559\times10^{-15}$$

$$P_f \approx 9.7559\times10^{-15}, \beta = 7.66$$

Example 5. Consider the same structure of example 2 with different external load $\mu_L = 20000, \sigma_L = 2000$. Calculate the structural reliability.

By the same procedure used in example 2, one can obtain the following results:

For L=20000, the ANSYS PROGRAMS and the corresponding results are listed in the appendixes: E,E1,E2,E3,E4,E5,F,F1,F2,F3,F4,F5,F6,G,G1,G2,G3,G4,and G5.

$$M_3 = R_3^{+} - S_3\,(tesnion) = R_3 - 14142 = R_3 - 0.7071L$$

$$M_{5\,(3)} = R_5^{-} + S_5\,(compression) = R_5 - 42426 = R_5 - 0.7071R_3$$

$$M_{6\,(3)} = R_6^{-} + S_6\,(compression) = R_6 - 42426 = R_6 - 0.7071R_3$$

$$M_4 = R_4^{-} + S_4\,(compression) = R_4 - 14142 = R_4 - 0.7071L$$

$$M_{1\,(4)} = R_1^{+} - S_1\,(tesnion) = R_1 - 42426 = R_1 - 0.7071R_4$$

$$M_{2(4)} = R_2^{\,+} - S_2(tesnion) = R_2 - 42426 = R_2 - 0.7071R_4$$

For the normalization: $Z_i = \dfrac{X_i - \mu_{X_i}}{\sigma_{X_i}}, X_i = \sigma_{X_i} Z_i + \mu_{X_i}$

$R_i = \sigma_{R_i} Z_i + \mu_{R_i} = 6000Z_i + 60000, \ (i = 1, \ldots, 6)$.

$L = \sigma_L Z_7 + \mu_L = 2000Z_7 + 20000$

$M_3 = R_3 - 0.7071L = (6000Z_3 + 60000) - 0.7071\,(2000Z_7 + 20000)$

$\Rightarrow 7.44 + 0.9733Z_3 - 0.2294Z_7$

$M_{5(3)} = R_5 - 0.7071R_3 = (6000Z_5 + 60000) - 0.7071\,(6000Z_3 + 60000)$

$\Rightarrow 2.39 - 0.5773Z_3 + 0.8165Z_5$

$M_{6(3)} = R_6 - 0.7071R_3 = (6000Z_6 + 60000) - 0.7071\,(6000Z_3 + 60000)$

$\Rightarrow 2.39 - 0.5773Z_3 + 0.8165Z_6$

$M_4 = R_4 - 0.7071L = (6000Z_4 + 60000) - 0.7071\,(2000Z_7 + 20000)$

$\Rightarrow 7.44 + 0.9733Z_4 - 0.2294Z_7$

$M_{1(4)} = R_1 - 0.7071R_4 = (6000Z_1 + 60000) - 0.7071\,(6000Z_4 + 60000)$

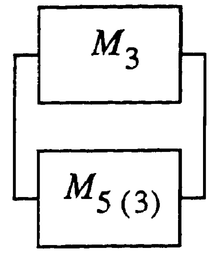
$\Rightarrow 2.39 + 0.8165Z_1 - 0.5773Z_4$

$$M_{2\,(4)} = R_2 - 0.7071R_4 = (6000Z_2 + 60000) - 0.7071\,(6000Z_4 + 60000)$$

$$\Rightarrow 2.39 + 0.8165Z_2 - 0.5773Z_4$$

$$M_3 = 7.44 + 0.9733Z_3 - 0.2294Z_7$$

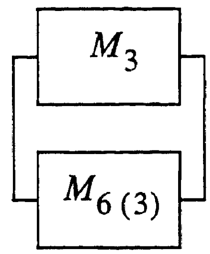
$$M_{5\,(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_5,\ \rho = -0.5619$$



$$\Phi_2\,(-7.44, -2.39; -0.5619) = 0.288352\times10^{-28},\ \beta = 11.18$$

$$M_3 = 7.44 + 0.9733Z_3 - 0.2294Z_7$$

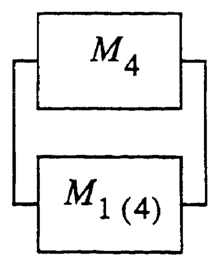
$$M_{6\,(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_6,\ \rho = -0.5619$$



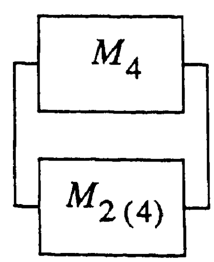
$$\Phi_2\,(-7.44, -2.39; -0.5619) = 0.288352\times10^{-28},\ \beta = 11.18$$

$$M_4 = 7.44 + 0.9733Z_4 - 0.2294Z_7$$

$$M_{1\,(4)} = 2.39 + 0.8165Z_4 - 0.5773Z_7,\ \rho = -0.5619$$



$$\Phi_2\,(-7.44, -2.39; -0.5619) = 0.288352\times10^{-28},\ \beta = 11.18$$

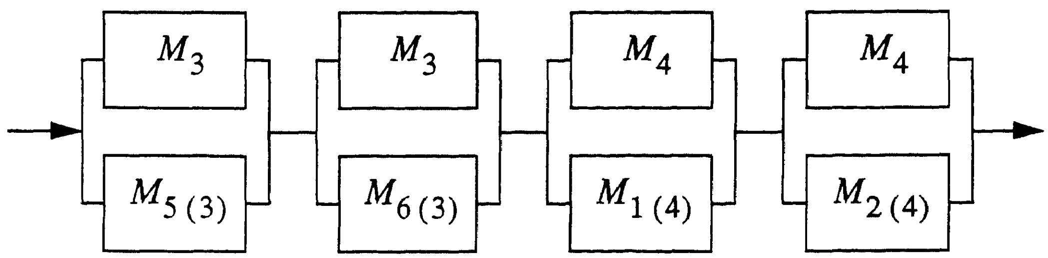


$$M_4 = 7.44 + 0.9733Z_4 - 0.2294Z_7$$

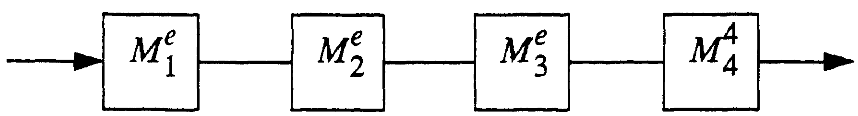
$$M_{2(4)} = 2.39 + 0.8165Z_2 - 0.5773Z_4,\ \rho = -0.5619$$

$$\Phi_2(-7.44, -2.39; -0.5619) = 0.288352 \times 10^{-28},\ \beta = 11.18$$

The reliability is modelled as a series system:



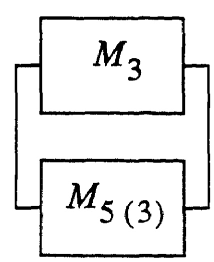
The equivalent series system is



The equivalent safety margins are calculated as follows:

$$M_3 = 7.44 + 0.9733Z_3 - 0.2294Z_7$$

$$M_{5(3)} = 2.39 - 0.5773Z_3 + 0.8165Z_5,\ \rho = -0.5619$$



$$\Phi_2(-7.44, -2.39; -0.5619) = 0.288352 \times 10^{-28},\ \beta = 11.18$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -7.44 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9733 & 0 & -0.2294 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -7.54 \\ -2.33 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.54, -2.33; -0.56)) = -\Phi^{-1}(1.4046 \times 10^{-29}) = 11.25$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -7.44 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9733 & 0 & -0.2294 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7.44 \\ -2.47 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.44, -2.47; -0.56)) = -\Phi^{-1}(1.3093 \times 10^{-29}) = 11.25$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -7.44 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9733 & 0 & -0.2294 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -7.42 \\ -2.39 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.42, -2.39; -0.56)) = -\Phi^{-1}(3.7409 \times 10^{-29}) = 11.16$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_1}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.25-11.18}{0.1} = 0.7$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_2}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.25-11.18}{0.1} = 0.7$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_3}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.16-11.18}{0.1} = -0.2$$

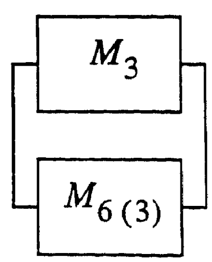
$$\begin{aligned}\bar{u}^e &= (u_1^e, u_2^e, u_3^e) \\ &= \left(\frac{0.7}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}, \frac{0.7}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}, \frac{-0.2}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}\right) \\ &= (0.6931, 0.6931, -0.198)\end{aligned}$$

Therefore,

$$M_1^e = 11.18 + 0.6931 Z_3 + 0.6931 Z_5 - 0.198 Z_7$$

$$M_3 = 7.44 + 0.9733 Z_3 - 0.2294 Z_7$$

$$M_{6(3)} = 2.39 - 0.5773 Z_3 + 0.8165 Z_6, \ \rho = -0.5619$$



$$\Phi_2(-7.44, -2.39; -0.5619) = 0.288352 \times 10^{-28}, \ \beta = 11.18$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -7.44 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9733 & 0 & -0.2294 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -7.54 \\ -2.33 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.54, -2.33; -0.56)) = -\Phi^{-1}(1.4046 \times 10^{-29}) = 11.25$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -7.44 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9733 & 0 & -0.2294 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0.1 \\ 0 \end{bmatrix} = \begin{bmatrix} -7.44 \\ -2.47 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.44, -2.47; -0.56)) = -\Phi^{-1}(1.3093 \times 10^{-29}) = 11.25$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix} -7.44 \\ -2.39 \end{bmatrix} - \begin{bmatrix} 0.9733 & 0 & -0.2294 \\ -0.5773 & 0.8165 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix} = \begin{bmatrix} -7.42 \\ -2.39 \end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.42, -2.39; -0.56)) = -\Phi^{-1}(3.7409 \times 10^{-29}) = 11.16$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_1}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{11.25 - 11.18}{0.1} = 0.7$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_2}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{11.25 - 11.18}{0.1} = 0.7$$

$$\left.\frac{\partial \beta_f}{\partial \varepsilon_3}\right|_{\bar{\varepsilon} = \bar{0}} \approx \frac{11.16 - 11.18}{0.1} = -0.2$$

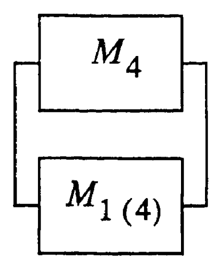
$$\begin{aligned}\bar{u}^e &= (u_1^e, u_2^e, u_3^e)\\ &= \left(\frac{0.7}{\sqrt{0.7^2 + 0.7^2 + (-0.2)^2}}, \frac{0.7}{\sqrt{0.7^2 + 0.7^2 + (-0.2)^2}}, \frac{-0.2}{\sqrt{0.7^2 + 0.7^2 + (-0.2)^2}}\right)\\ &= (0.6931, 0.6931, -0.198)\end{aligned}$$

Therefore,

$$M_2^e = 11.18 + 0.6931 Z_3 + 0.6931 Z_6 - 0.198 Z_7$$

$$M_4 = 744 + 0.9733 Z_4 - 0.2294 Z_7$$

$$M_{1(4)} = 2.39 + 0.8165 Z_1 - 0.5773 Z_4, \ \rho = -0.5619$$



$$\Phi_2(-7.44, -2.39; -0.5619) = 2.88352 \times 10^{-29}, \ \beta = 11.18$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-7.44\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9733 & -0.2294\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0.1\\0\\0\end{bmatrix} = \begin{bmatrix}-7.44\\-2.47\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.44, -2.47; -0.56)) = -\Phi^{-1}(1.30927\times10^{-29}) = 11.25$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-7.44\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9733 & -0.2294\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0\\0.1\\0\end{bmatrix} = \begin{bmatrix}-7.54\\-2.33\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.54, -2.33; -0.52)) = -\Phi^{-1}(1.4046\times10^{-29}) = 11.25$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-7.44\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9733 & -0.2294\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.1\end{bmatrix} = \begin{bmatrix}-7.42\\-2.39\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.42, -2.39; -0.56)) = -\Phi^{-1}(3.7409\times10^{-29}) = 11.16$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_1}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.25-11.18}{0.1} = 0.7$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_2}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.25-11.18}{0.1} = 0.7$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_3}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.16-11.18}{0.1} = -0.2$$

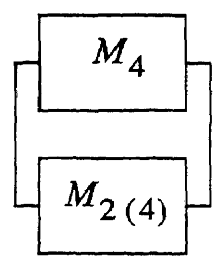
$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

$$= \left(\frac{0.7}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}, \frac{0.7}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}, \frac{-0.2}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}\right)$$

$$= (0.6931, 0.6931, -0.198)$$

Therefore, $M_3^e = 11.18 + 0.6931Z_3 + 0.6931Z_6 - 0.198Z_7$

$$M_4 = 7.44 + 0.9733Z_4 - 0.2294Z_7$$



$$M_{2(4)} = 2.39 + 0.8165Z_2 - 0.5773Z_4,\ \rho = -0.5619$$

$$\Phi_2(-.44, -2.39; -0.5619) = 2.88352\times10^{-29},\ \beta = 11.18$$

$$\bar{\varepsilon}_1 = (0.1, 0, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta}-\bar{u}\bar{\varepsilon} = \begin{bmatrix}-7.44\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9733 & -0.2294\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0.1\\0\\0\end{bmatrix} = \begin{bmatrix}-7.44\\-2.47\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.44, -2.47; -0.56)) = -\Phi^{-1}(1.3093\times10^{-29}) = 11.25$$

$$\bar{\varepsilon}_2 = (0, 0.1, 0)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix}-7.44\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9733 & -0.2294\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0\\0.1\\0\end{bmatrix} = \begin{bmatrix}-7.54\\-2.33\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.54, -2.33; -0.56)) = -\Phi^{-1}(1.4046\times10^{-29}) = 11.25$$

$$\bar{\varepsilon}_3 = (0, 0, 0.1)$$

$$-\beta(\bar{\varepsilon}) = -\bar{\beta} - \bar{u}\bar{\varepsilon} = \begin{bmatrix}-7.44\\-2.39\end{bmatrix} - \begin{bmatrix}0 & 0.9733 & -0.2294\\0.8165 & -0.5773 & 0\end{bmatrix}\begin{bmatrix}0\\0\\0.1\end{bmatrix} = \begin{bmatrix}-7.42\\-2.39\end{bmatrix}$$

$$\beta_f(\bar{\varepsilon}) = -\Phi^{-1}(\Phi_2(-7.42, -2.39; -0.56)) = -\Phi^{-1}(3.7409\times10^{-29}) = 11.16$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_1}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.25 - 11.18}{0.1} = 0.7$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_2}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.25 - 11.18}{0.1} = 0.7$$

$$\left.\frac{\partial\beta_f}{\partial\varepsilon_3}\right|_{\bar{\varepsilon}=\bar{0}} \approx \frac{11.16-11.18}{0.1} = -0.2$$

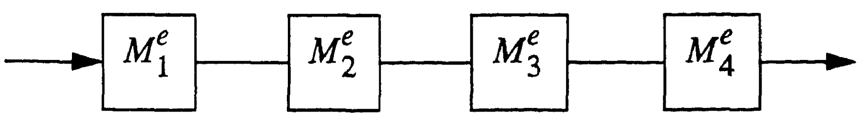
$$\bar{u}^e = (u_1^e, u_2^e, u_3^e)$$

$$= \left(\frac{0.7}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}, \frac{0.7}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}, \frac{-0.2}{\sqrt{0.7^2+0.7^2+(-0.2)^2}}\right)$$

$$= (0.6931, 0.6931, -0.198)$$

Therefore, $M_4^e = 11.18 + 0.6931Z_2 + 0.6931Z_4 - 0.198Z_7$

The Ditlevsen bounds for the probability of failure of the series system are:



$M_1^e = 11.18 + 0.6931Z_3 + 0.6931Z_5 - 0.198Z_7$, $\rho_{12} = 0.5196$,

$\rho_{13} = 0.0392$

$M_2^e = 11.18 + 0.6931Z_3 + 0.6931Z_6 - 0.198Z_7$, $\rho_{14} = 0.0392$,

$\rho_{23} = 0.0392$

$M_3^e = 11.18 + 0.6931Z_1 + 0.6931Z_4 - 0.198Z_7$, $\rho_{24} = 0.0392$,

$\rho_{35} = 0.5196$

$$M_4^e = 11.18 + 0.6931 Z_2 + 0.6931 Z_4 - 0.198 Z_7$$

upper bound: $P_f \le \sum_{i=1}^{n} \Phi(-\beta_i) - \sum_{i=2, j<i}^{n} max \Phi_2(-\beta_i, -\beta_j; \rho)$

$$P_f \le \Phi(-\beta_1) + \Phi(-\beta_2) + \Phi(-\beta_3) + \Phi(-\beta_4) - max[\Phi_2(-\beta_2, -\beta_1; \rho)]$$

$$-max[\Phi_2(-\beta_3, -\beta_2; \rho), \Phi_2(-\beta_3, -\beta_1; \rho)]$$

$$-max[\Phi(-\beta_4, -\beta_1; \rho), \Phi(-\beta_4, -\beta_2; \rho), \Phi(-\beta_4, -\beta_3; \rho)]$$

$$= 4 \times 2.5545 \times 10^{-29} - 6.29596 \times 10^{-39} - 7.86105 \times 10^{-56} - 6.29596 \times 10^{-39}$$

$$= 1.0218 \times 10^{-28}$$

lower bound: $P_f \ge \Phi(-\beta_1) + \sum_{i=2}^{n} max \left[\Phi(-\beta_i) - \sum_{j=1}^{i-1} \Phi_2(-\beta_i, -\beta_j; \rho), 0 \right]$

$$P_f \ge \Phi(-\beta_1) + max[\Phi(-\beta_2) - \Phi_2(-\beta_2, -\beta_1; \rho), 0]$$

$$+ max[\Phi(-\beta_3) - \Phi_2(-\beta_3, -\beta_1; \rho) - \Phi_2(-\beta_3, -\beta_2; \rho), 0] +$$

$$max[\Phi(-\beta_4) - \Phi_2(-\beta_4, -\beta_1; \rho) - \Phi_2(-\beta_4, -\beta_2; \rho) - \Phi_2(-\beta_4, -\beta_3; \rho), 0]$$

$$= 2.5545 \times 10^{-29} + 2.5545 \times 10^{-29} - 6.29596 \times 10^{-39} + 2.5545 \times 10^{-29}$$

$$- 7.86105 \times 10^{-56} - 7.86105 \times 10^{-56} + 2.5545 \times 10^{-29} - 7.86105 \times 10^{-56}$$

$$- 7.86105 \times 10^{-56} - 6.29596 \times 10^{-39} = 1.0218 \times 10^{-28}$$

$P_f \approx 1.0218\times10^{-28}$, $\beta = 11.07$

Example 6. Consider the effects of loads on structures: (1) probabilities vs. loads (2) dafe-indices vs. loads (3) probabilities vs. safety-factors.

$$\text{S.F.} = \mathit{safety\ factor} = \frac{\mathit{ultimate\ load}}{\mathit{allowable\ load}}$$

(a).

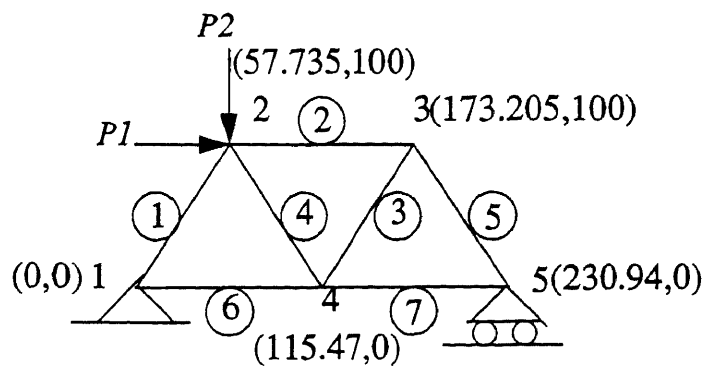


figure a

For $p = p_1 = p_2 = 10000$; $P_f = \Phi(-\beta) = \Phi(-7.95) = 9.823\times10^{-16}$;

$$\text{S.F.} = \frac{60000}{11830} = 5.07$$

For $p = p_1 = p_2 = 20000$; $P_f = \Phi(-\beta) = \Phi(-5.82) = 2.951\times10^{-9}$;

$$\text{S.F.} = \frac{60000}{23660} = 2.54$$

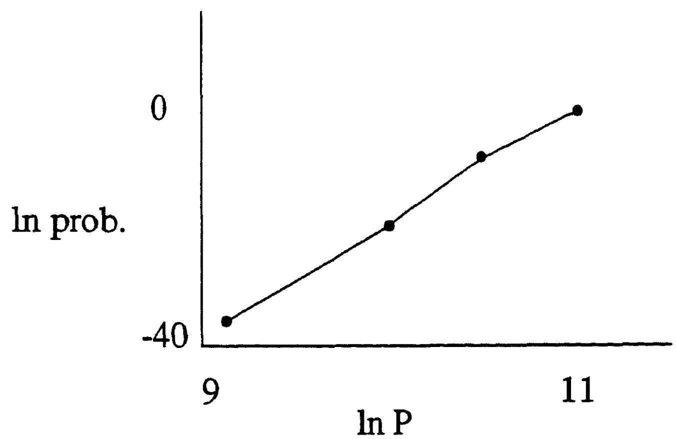
For $p = p_1 = p_2 = 30000$; $P_f = \Phi(-\beta) = \Phi(-3.75) = 8.837\times10^{-5}$;

$$\text{S.F.} = \frac{60000}{35490} = 1.69$$

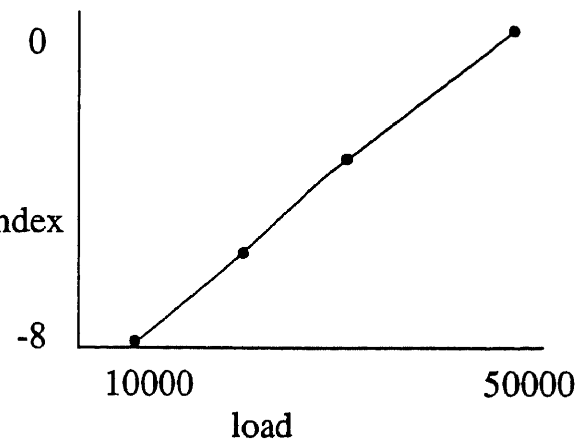
For $p = p_1 = p_2 = 50000$; $P_f = \Phi(-\beta) = \Phi(-0.11) = 0.4562$

$$\text{S.F.} = \frac{60000}{59150} = 1.01$$

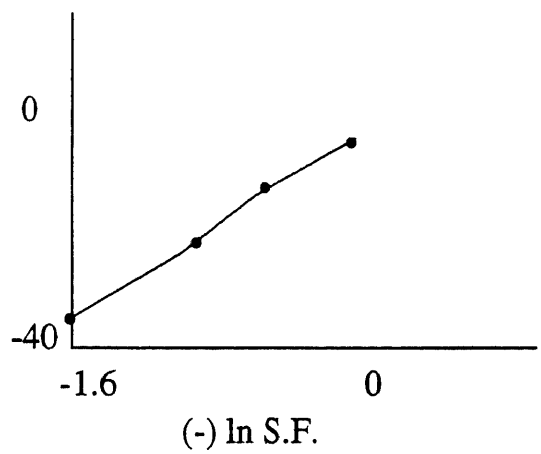
(1) ln prob. vs. ln load.



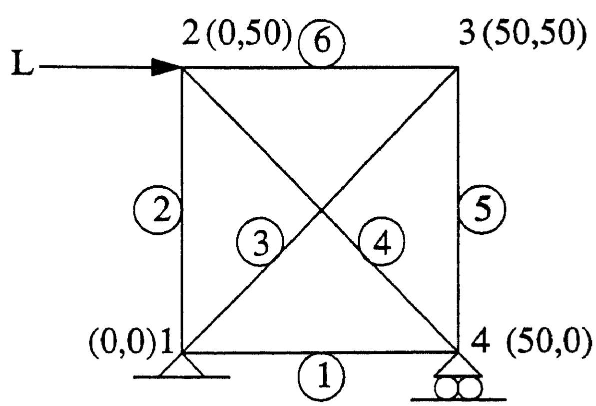
(2) (-) safety index vs. load.



(3) ln prob. vs. (-) ln S.F.



(b).



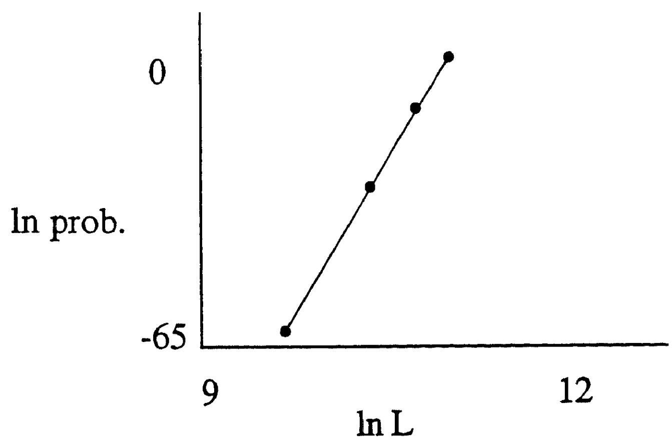
For L = 20000; $P_f = \Phi(-\beta) = \Phi(-11.07) = 1.02\times10^{-28}$; S.F. = 4.24

For L = 40000; $P_f = \Phi(-\beta) = \Phi(-7.66) = 9.756\times10^{-15}$; S.F. = 2.12

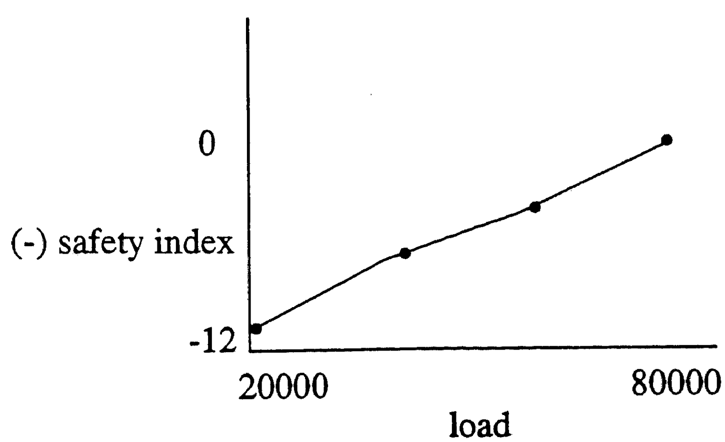
For L = 60000; $P_f = \Phi(-\beta) = \Phi(-4.85) = 0.6138\times10^{-6}$; S.F. = 1.41

For L = 80000; $P_f = \Phi(-\beta) = \Phi(-0.52) = 0.3006$; S.F. = 1.06

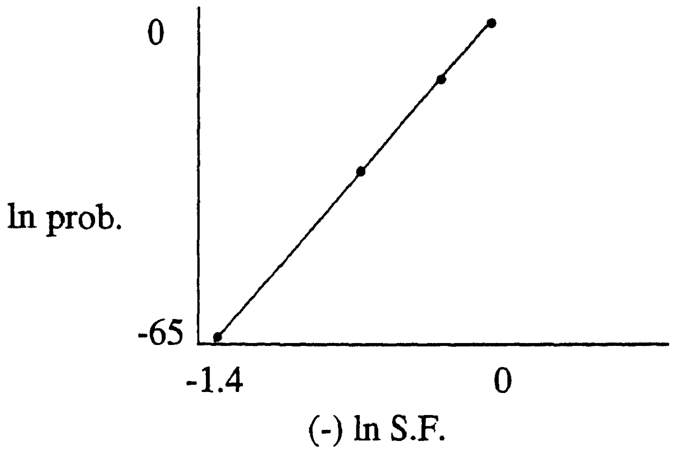
(1) ln prob. vs. ln load.



(2) (-) safety index vs. load.



(3) ln prob. vs. (-) ln S.F.



APPENDIX A ANSYS PROGRAM AND FORCE DISTRIBUTIONS

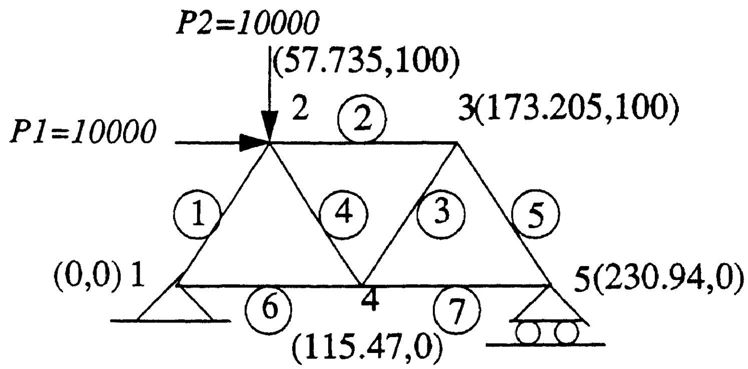


figure a

```
/PREP7
KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,57.735,100 $N,3,173.205,100
N,4,115.47 $N,5,230.94 $E,1,2 $E,2,3 $E,3,4 $E,2,4 $E,3,5 $E,1,4 $E,4,5 $D,1,ALL,0
D,5,UY,0 $F,2,FX,10000 $F,2,FY,-10000 $ITER,1,1,1 $AFWRITE $FINISH
$/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE $FINISH
```

FOREC DISTRIBUTIONS:

	ELEMENT	FORCE
1	-3660.3	compression
2	-7886.8	compression
3	7886.8	tension
4	-7886.8	compression
5	-7886.8	compression

6	11830	tension
7	3943.4	tension

APPENDIX A1ANSYS PROGRAM AND FORCE DISTRIBUTIONS

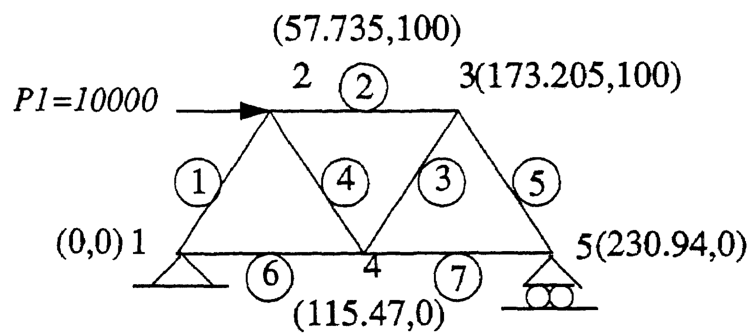


figure a1

/PREP7

KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,57.735,100

$N,3,173.205,100

N,4,115.47 $N,5,230.94 $E,1,2 $E,2,3 $E,3,4 $E,2,4 $E,3,5 $E,1,4 $E,4,5 $D,1,ALL,0

D,5,UY,0 $F,2,FX,10000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH

/POST1 $SET $PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	5000	tension

2	-5000	compression
3	5000	tension
4	-5000	compression
5	-5000	compression
6	7500	tension
7	2500	tension

APPENDIX A2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

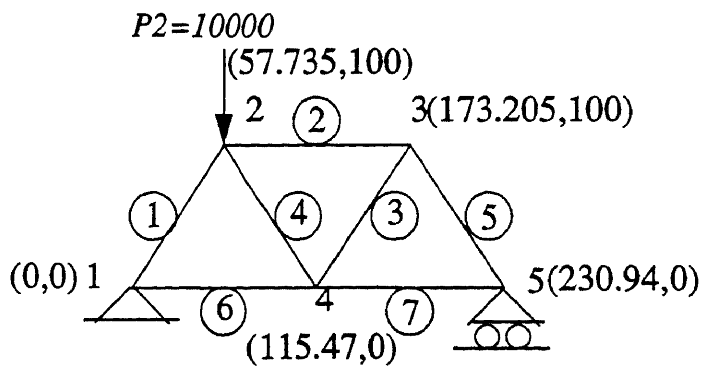


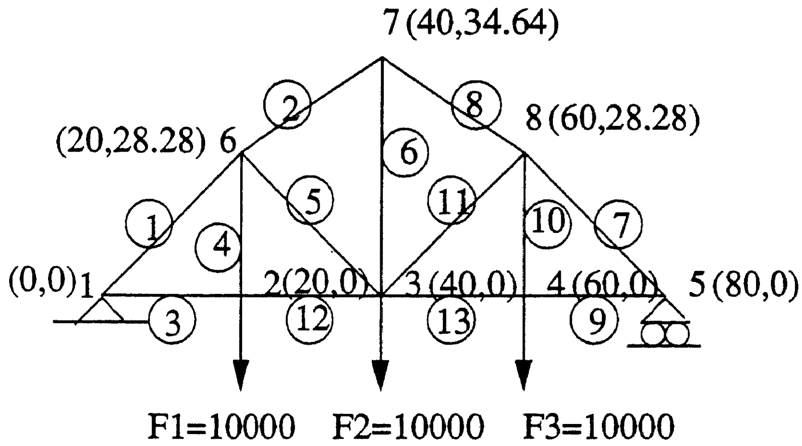
figure a2

```
/PREP7
KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,57.735,100
$N,3,173.205,100 N,4,115.47 $ N,5,230.94 $E,1,2 $E,2,3 $E,3,4 $E,2,4 $E,3,5 $E,1,4
$E,4,5 $D,1,ALL,0 D,5,UY,0 $F,2,FY,-10000 $ITER,1,1,1 $AFWRITE $FINISH $/
INPUT,27 $FINISH
/POST1 $SET $PRDISP $NFORCE $FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-8660.3	compression
2	-2886.8	compression
3	2886.8	tension
4	-2886.8	compression
5	-2886.8	compression
6	4330.1	tension
7	1443.4	tension

APPENDIX B ANSYS PROGRAM AND FORCE DISTRIBUTIONS



(Figure b)

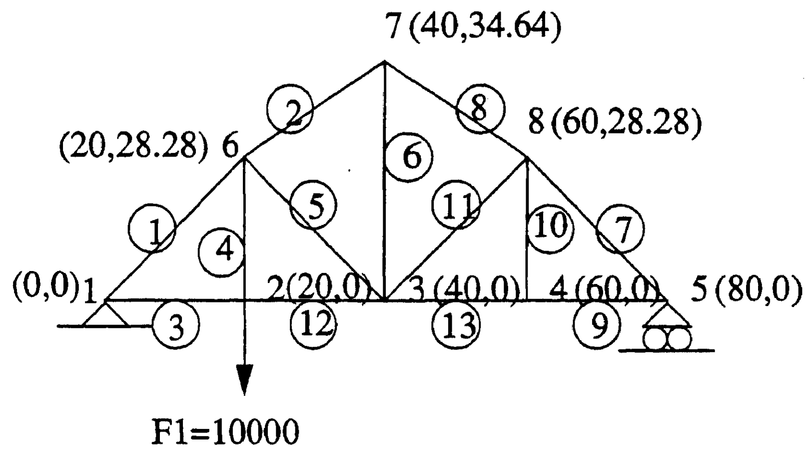
```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,20 $N,3,40 $N,4,60
N,5,80 $N,6,20,28.2843 $N,7,40,34.641 $N,8,60,28.2843 $E,1,6 $E,6,7 $E,1,2
E,2,6 $E,3,6 $E,3,7 $E,5,8 $E,7,8 $E,4,5 $E,4,8 $E,3,8 $E,2,3 $E,3,4 $D,1,ALL,0
D,5,UY,0 $F,2,FY,-10000 $F,3,FY,-10000 $F,4,FY,-10000 $ITER,1,1,1 $AFWRITE
FINISH $/INPUT,27 $FINISH $/POST1 $PRDISP $NFORCE $FINISH
```

FORCE DISTRIBUTIONS

	ELEMENT	FORCE
1	-18371	compression
2	-12116	compression
3	10607	tension

4	10000	tension
5	1628.9	tension
6	7340.1	tension
7	-18371	compression
8	-12116	compression
9	10607	tension
10	10000	tension
11	1628.9	tension
12	10607	tension
13	10607	tension

APPENDIX B1 ANSYS PROGRAM AND FOREC DISTRIBUTION



(Figure b1)

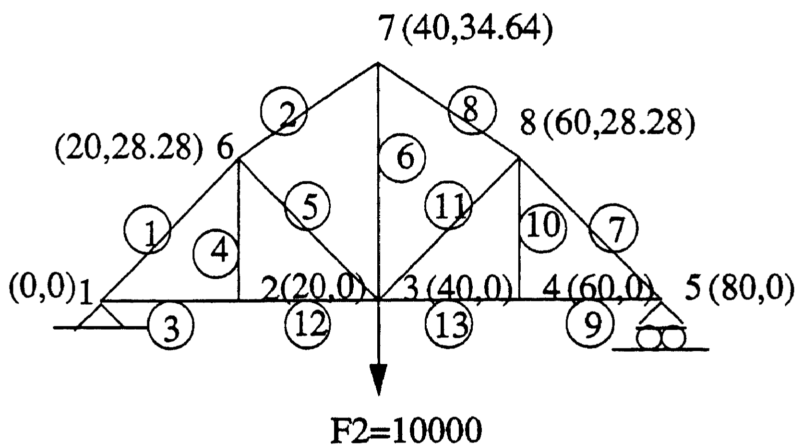
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,20 $N,3,40 $N,4,60 $N,5,80 N,6,20,28.2843 $N,7,40,34.641 $N,8,60,28.2843 $E,1,6 $E,6,7 $E,1,2 $E,2,6 $E,3,6 E,3,7 $E,5,8 $E,7,8 $E,4,5 $E,4,8 $E,3,8 $E,2,3 $E,3,4 $D,1,ALL,0 $D,5,UY,0 F,2,FY,-10000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 SET $PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-9185.6	compression
2	-3029.1	compression
3	5303.3	tension
4	10000	tension
5	-4185.6	compression
6	1835	tension
7	-3061.9	compression

8	-3029.1	compression
9	1767.8	tension
10	0	
11	1938.1	tension
12	5303.3	tension
13	1767.8	tension

APPENDIX B2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



(Figure b2)

/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,20 $N,3,40 $N,4,60

N,5,80 $N,6,20,28.2843 $N,7,40,34.641 $N,8,60,28.2843 $E,1,6 $E,6,7 $E,1,2 $E,2,6

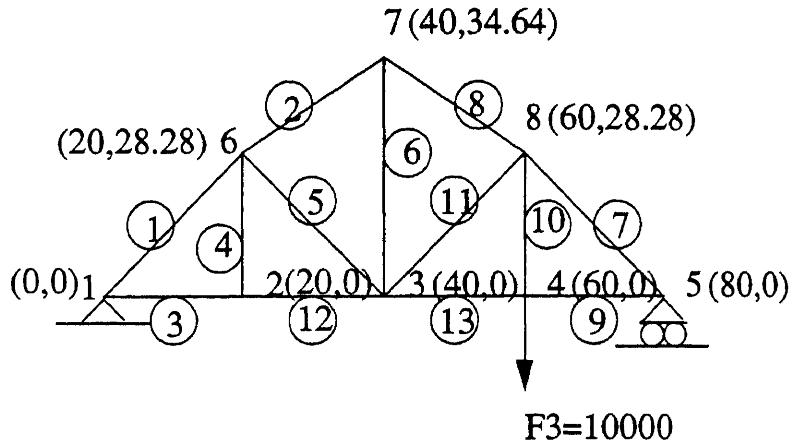
E,3,6 $E,3,7 $E,5,8 $E,7,8 $E,4,5 $E,4,8 $E,3,8 $E,2,3 $E,3,4 $D,1,ALL,0 $D,5,UY,0

F,3,FY,-10000 $ITER,1,1,1 $AFWRITE $FINISH %/INPUT,27 $FINISH $/POST1 SET $PRDISP $NFORCE $FINIS

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-6123.7	compression
2	-6058.1	compression
3	3535.5	tension
4	0	
5	3876.3	tension
6	3670	tension
7	-6123.7	compression
8	-6058.1	compression
9	3535.5	tension
10	0	
11	3876.3	tension
12	3535.3	tesnion
13	3535.3	tension

APPENDIX B3 ANSYS PROGRAM AND FORCEDISTRIBUTIONS



(Figure b3)

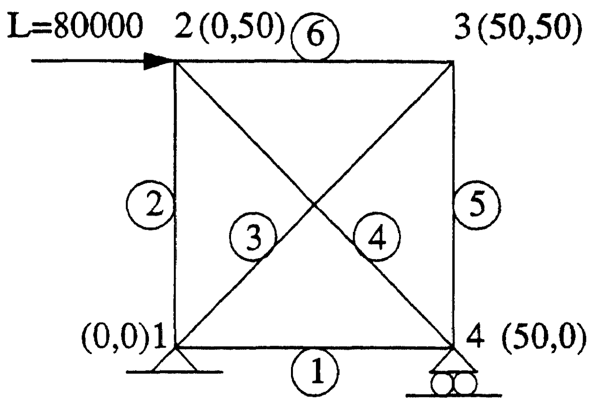
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,20 $N,3,40 $N,4,60
N,5,80 $N,6,20,28.2843 $N,7,40,34.641 $N,8,60,28.2843 $E,1,6 $E,6,7 $E,1,2 $E,2,6
E,3,6 $E,3,7 $E,5,8 $E,7,8 $E,4,5 $E,4,8 $E,3,8 $E,2,3 $E,3,4 $D,1,ALL,0 $D,5,UY,0
F,4,FY,-10000 4ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET
PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-3061.9	compression
2	-3029.1	compression
3	1767.8	tension

4	0	
5	1938.1	tension
6	1835	tension
7	-9185.6	compression
8	-3029.1	compression
9	5303.3	tension
10	10000	tension
11	-4185.6	compression
12	1767.8	tension
13	5303.3	tension

APPENDIX C ANSYS PROGRAM AND FORCE DISTRIBUTIONS

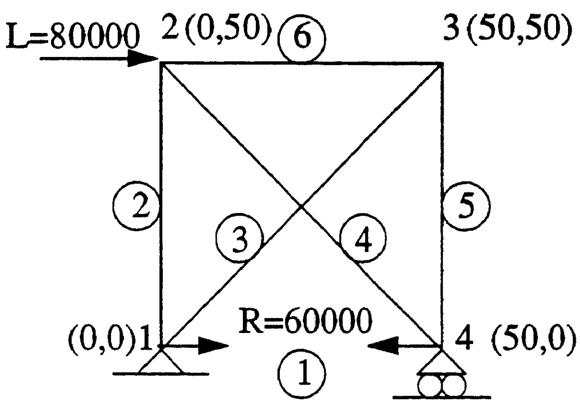



```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000
 ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $NFORCE
FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	40000	tension
2	40000	tension
3	56569	tension
4	-56569	compression
5	-40000	compression
6	-40000	compression

APPENDIX C1 ANSYS PROGRAMS AND FORCE DISTRIBUTIONS

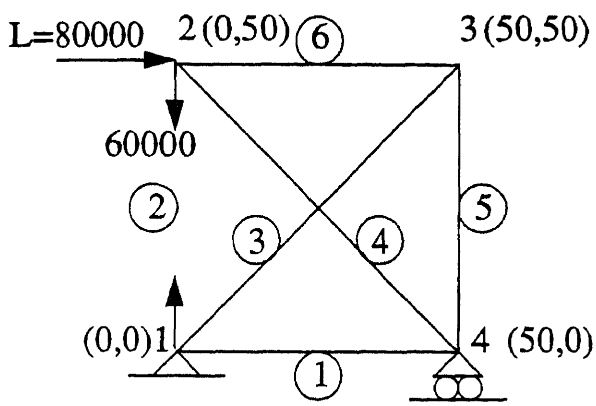


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 $F,1,FX,60000 F,4,FX,-60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
2	60000	tension
3	28284	tension
4	-84853	compression
5	-20000	compression
6	-20000	compression

APPENDIX C2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

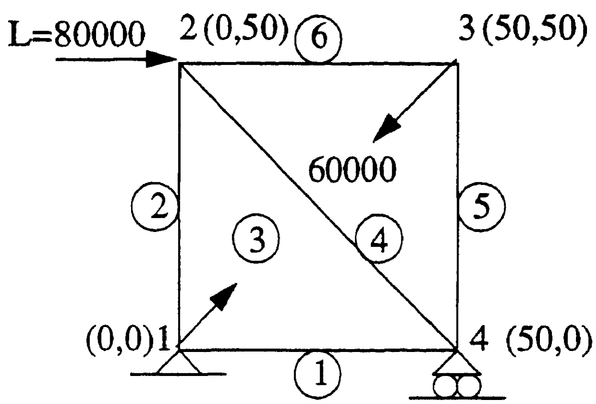


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,4 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 $F,2,FY,-60000 F,1,FY,60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	60000	tension
3	28284	tension
4	-84853	compression
5	-20000	compression
6	-20000	compression

APPENDIX C3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

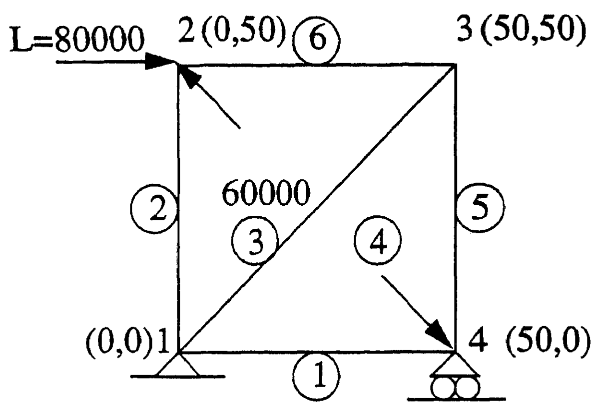


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 F,1,FX,42426.4
F,1,FY,42426.4 $F,3,FX,-42426.4 $F,3,FY,-42426.4 $ITER,1,1,1 $AFWRITE $FINISH
/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	37574	tension
2	37574	tension
4	-53137	compression
5	-42426	compression
6	-42426	compression

APPENDIX C4 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

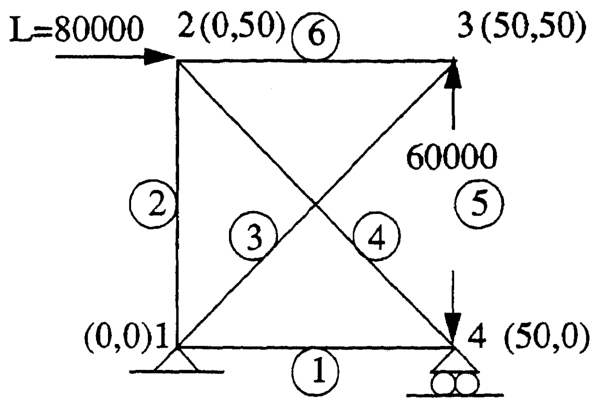



```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,37573.6
F,2,FY,42426.4 $F,4,FX,42426.4 $F,4,FY,-42426.4 $ITER,1,1,1 $AFWRITE $FINISH
/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE $FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	42426	tension
2	42426	tension
3	53137	tension
5	-37574	compression
6	-37574	compression

APPENDIX C5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,3E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0

E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 $F,3,FY,60000

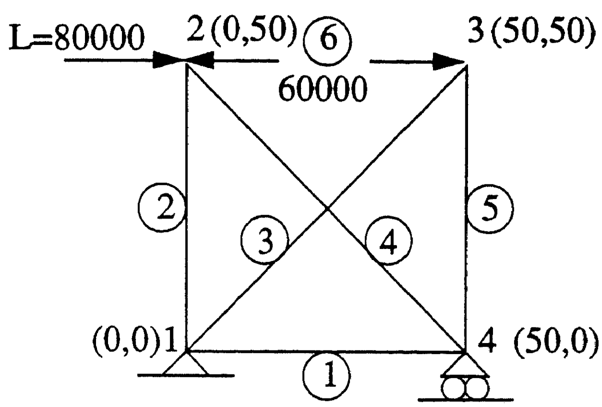
F,4,FY,-60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET

PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	20000	tension
2	20000	tension
3	84853	tension
4	-28284	compression
6	-60000	compression

APPENDIX C6 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,3E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0

$E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 F,3,FX,60000

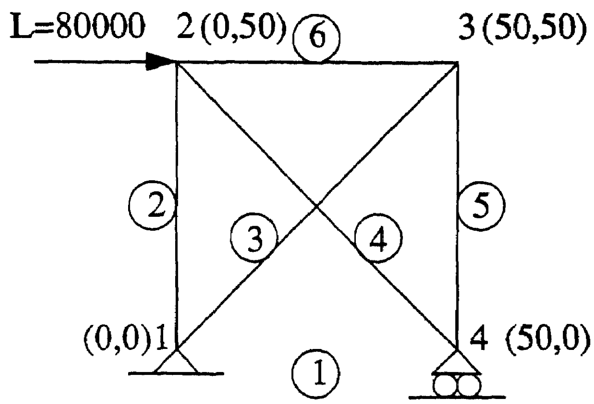
ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP

NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	20000	tension
2	20000	tension
3	84853	tension
4	-28284	compression
5	-60000	compression

APPENDIX D ANSYS PROGRAM AND FORCE DISTRIBUTIONS

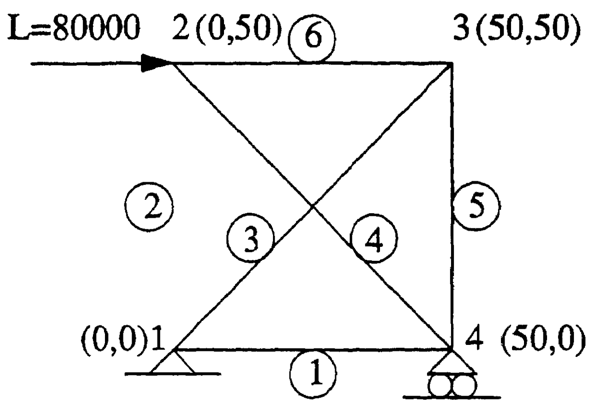


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000
ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP
NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
2	0	
3	113140	tension
4	0	
5	-80000	compression
6	-80000	compression

APPENDIX D1 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

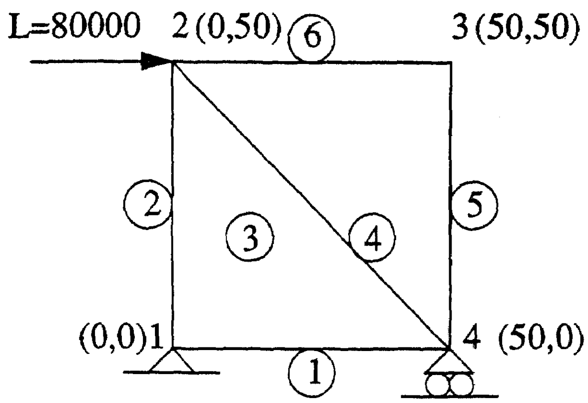



```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	0	
3	113140	tension
4	0	
5	-80000	compression
6	-80000	compression

APPENDIX D2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

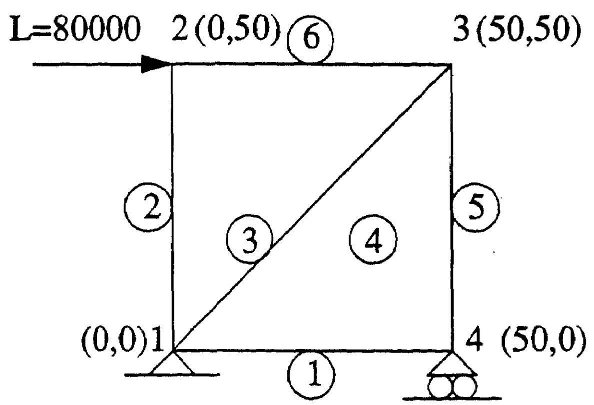


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,4 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 $ITER,1,1,1 AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	80000	tension
2	80000	tension
4	-113140	compression
5	0	
6	0	

APPENDIX D3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

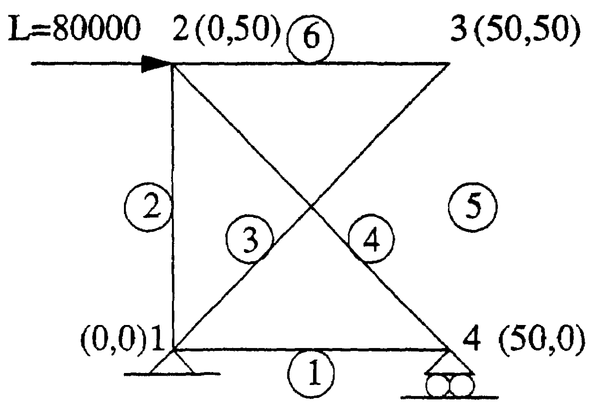



```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	0	
2	0	
3	113140	tension
5	-80000	compression
6	-80000	compression

APPENDIX D4 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

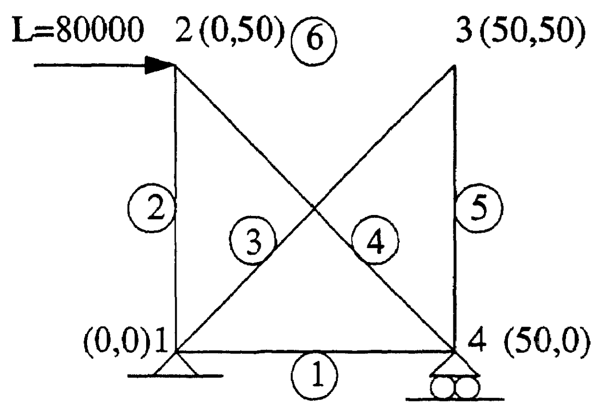


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	80000	tension
2	80000	tension
3	0	
4	-113140	compression
6	0	

APPENDIX D5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

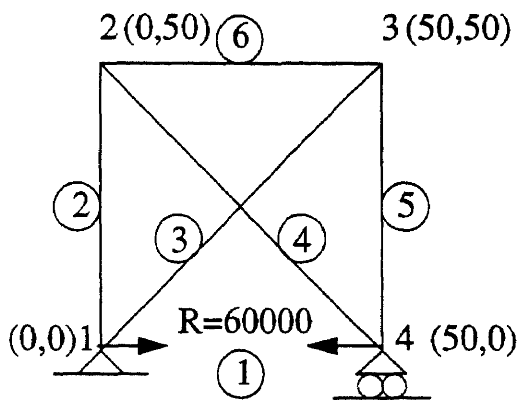


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,80000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	80000	tension
2	80000	tension
3	0	
4	-113140	compression
5	0	

APPENDIX E ANSYS PROGRAM AND FORCE DISTRIBUTIONS



/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0

E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,1,FX,60000 $F,4,FX,-60000

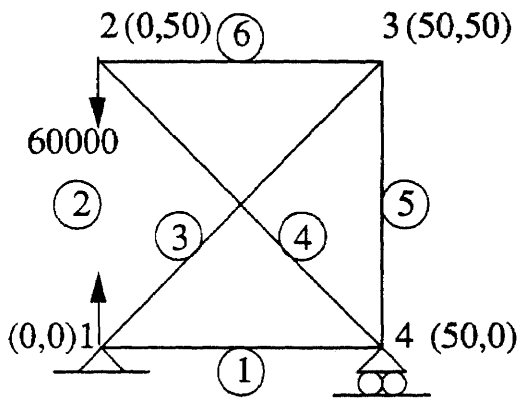
ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP

NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
2	60000	tension
3	-84853	compression
4	-84853	compression
5	60000	tension
6	60000	tension

APPENDIX E1 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

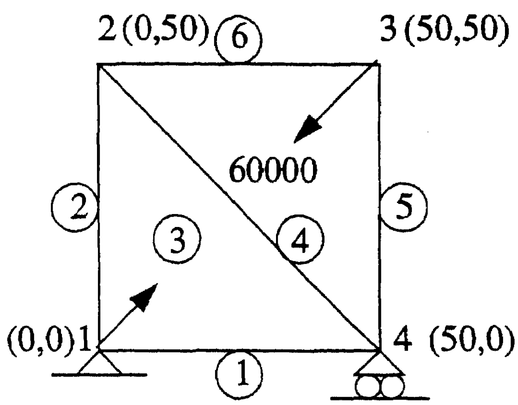


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FY,-60000 $F,1,FY,60000
ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP
NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	60000	tension
3	-84853	compression
4	-84853	compression
5	60000	tension
6	60000	tension

APPENDIX E2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

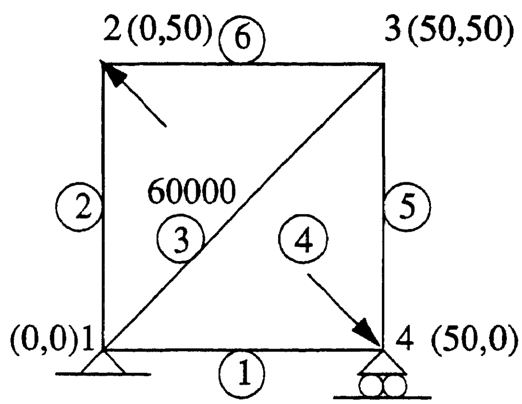


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,1,FX,42426.4
F,1,FY,42426.4 $F,3,FX,-42426.4 $F,3,FY,-42426.4 $ITER,1,1,1 $AFWRITE $FINISH
/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-42426	compression
2	-42426	compression
4	60000	tension
5	-42426	compression
6	-42426	compression

APPENDIX E3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

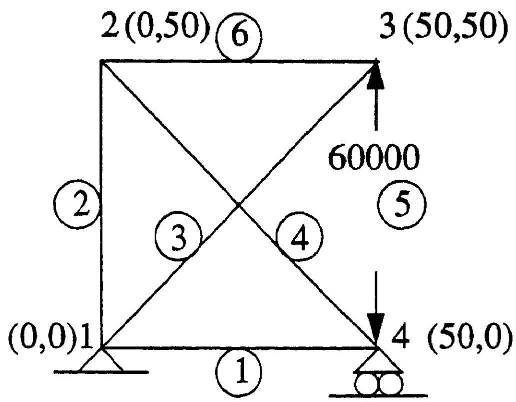


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,-42426.4
F,2,FY,42426.4 $F,4,FX,42426.4 $F,4,FY,-42426.4 $ITER,1,1,1 $AFWRITE $FINISH
/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	42426	tension
2	42426	tension
3	-60000	compression
5	42426	tension
6	42426	tension

APPENDIX E4 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

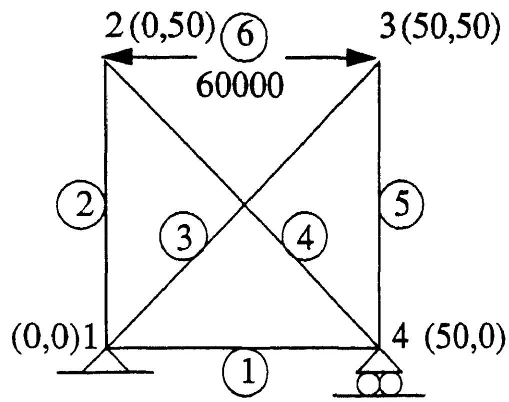


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,3,FY,60000 $F,4,FY,-60000 ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $%FINISH $/POST1 $SET $PRDISP NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-60000	compression
2	-60000	compression
3	84853	tension
4	84853	tension
6	-60000	compression

APPENDIX E5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

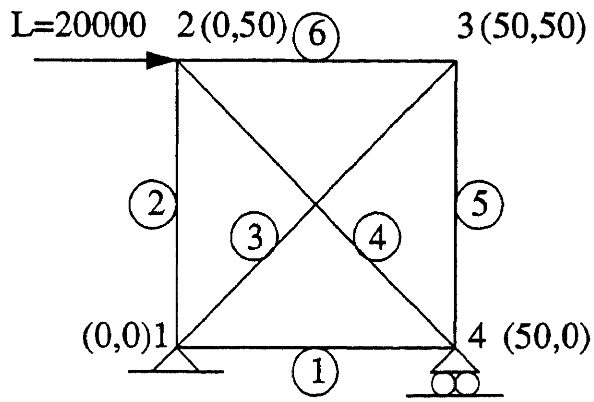



```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,-60000 $F,3,FX,60000
ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP
NFORCE $FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-60000	compression
2	-60000	compression
3	84853	tension
4	84853	tension
5	-60000	compression

APPENDIX F ANSYS PROGRAM AND FORCE DISTRIBUTIONS

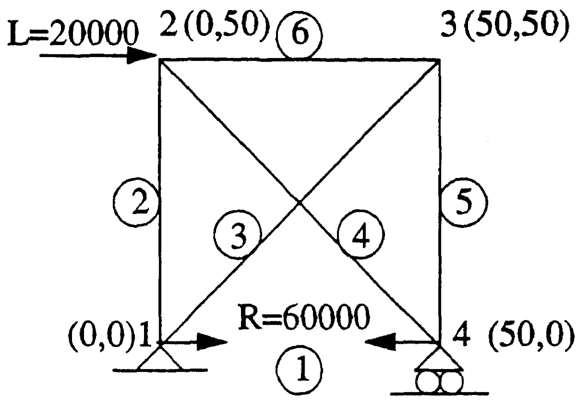


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000
ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP
NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	10000	tension
2	10000	tension
3	14142	tension
4	-14142	compression
5	-10000	compression
6	-10000	compression

APPENDIX F1 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

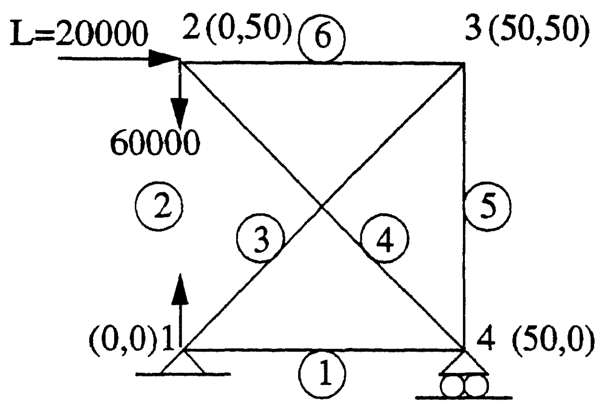


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $F,1,FX,60000 F,4,FX,-60000 $ITER,1,1,1 $AFWRITE$FINISH $/INPUT,27 $FINISH $/POST1 $SET PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
2	60000	tension
3	-56569	compression
4	-84853	compression
5	40000	tension
6	40000	tension

APPENDIX F2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

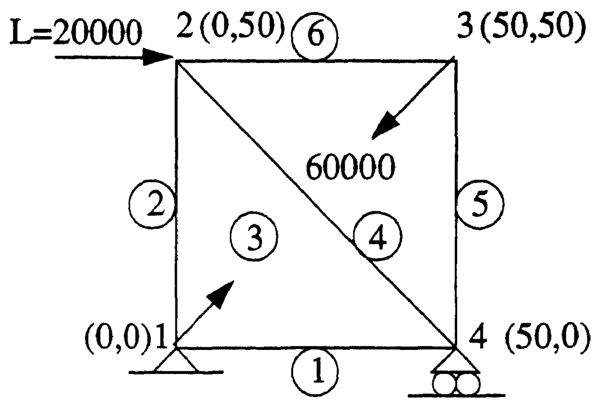


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,4 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $F,2,FY,-60000 F,1,FY,60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	60000	tension
3	-56569	compression
4	-84853	compression
5	40000	tension
6	40000	tension

APPENDIX F3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

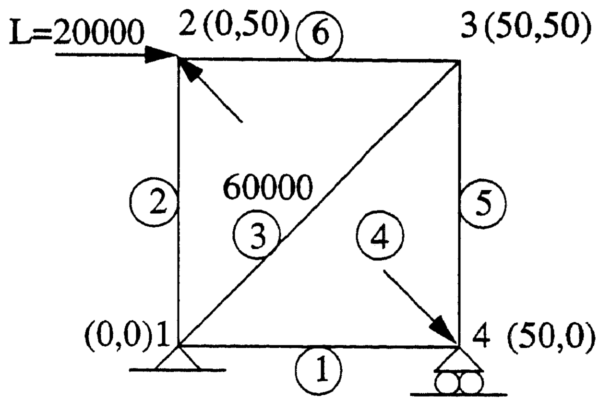


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,4 $E,1,2 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 F,1,FX,42426.4 F,1,FY,42426.4 $F,3,FX,-42426.4 $F,3,FY,-42426.4 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-22426	compression
2	-22426	compression
4	31716	tension
5	-42426	compression
6	-42426	compression

APPENDIX F4 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

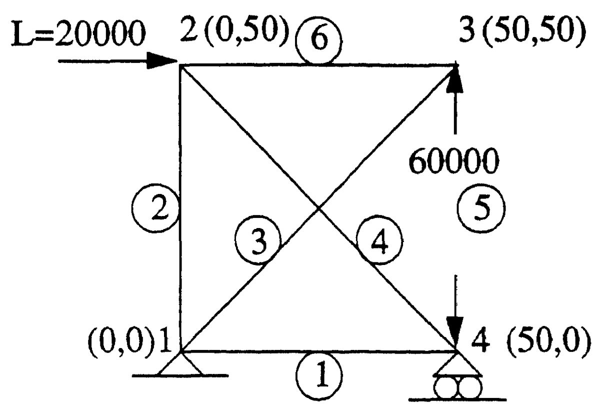



```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,-22426.4
F,2,FY,42426.4 $F,4,FX,42426.4 $F,4,FY,-42426.4 $ITER,1,1,1 $AFWRITE $FINISH
/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE $FINISH
```

FORCE DISTRIBUTION:

	ELEMENT	FORCE
1	42426	tension
2	42426	tension
3	-31716	compression
5	22426	tension
6	22426	tension

APPENDIX F5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

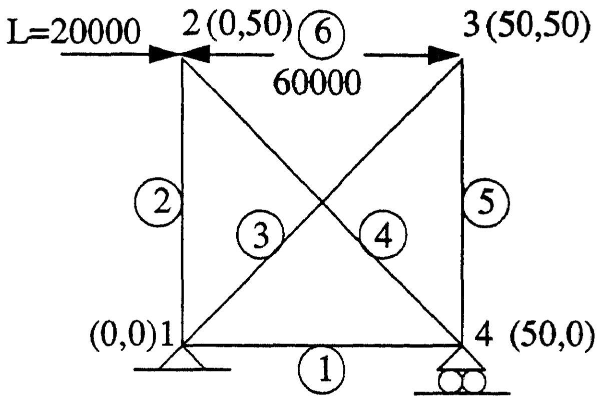


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $F,3,FY,60000 F,4,FY,-60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-40000	compression
2	-40000	compression
3	84853	tension
4	56569	tension
6	-60000	compression

APPENDIX F6 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0

E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,-40000 $F,3,FX,60000

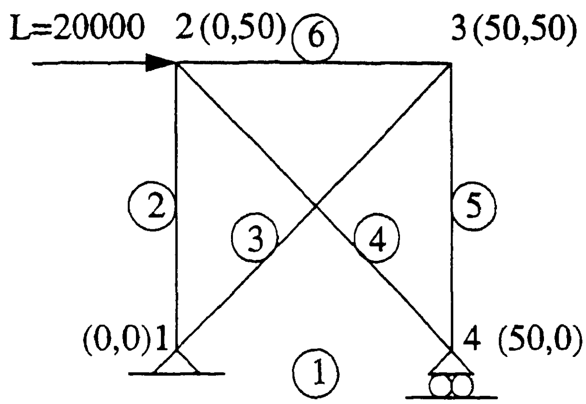
ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP

$NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-40000	compression
2	-40000	compression
3	84853	tension
4	56569	tension
5	-60000	compression

APPENDIX G ANSYS PROGRAM AND FORCE DISTRIBUTIONS



/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0

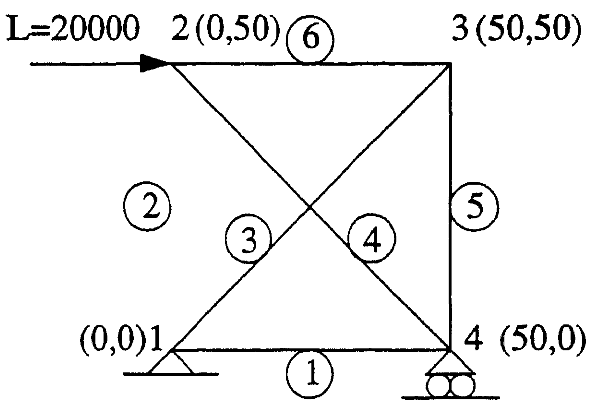
E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $ITER,1,1,1

AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
2	0	
3	28284	tension
4	0	
5	-20000	compression
6	-20000	compression

APPENDIX G1 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

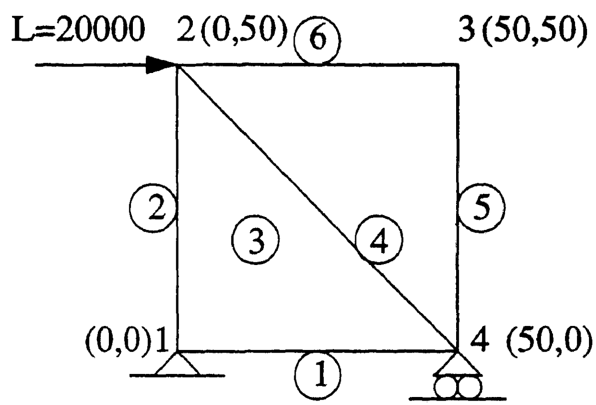


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTION:

	ELEMENT	FORCE
1	0	
3	28284	tension
4	0	
5	-20000	compression
6	-20000	compression

APPENDIX G2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

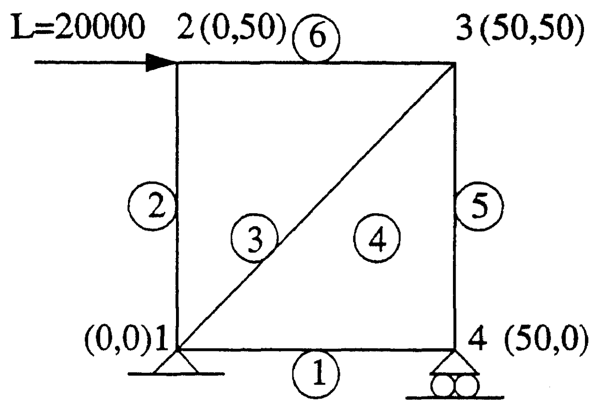



```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $ ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	20000	tension
2	20000	tension
4	-28284	compression
5	0	
6	0	

APPENDIX G3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

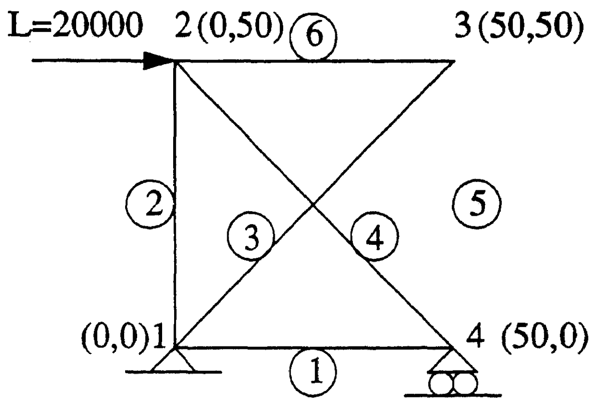



```
/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	0	
2	0	
3	28284	tension
5	-20000	compression
6	-20000	compression

APPENDIX G4 ANSYS PROGRAM AND FORCE DISTRIBUTRIONS



/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0

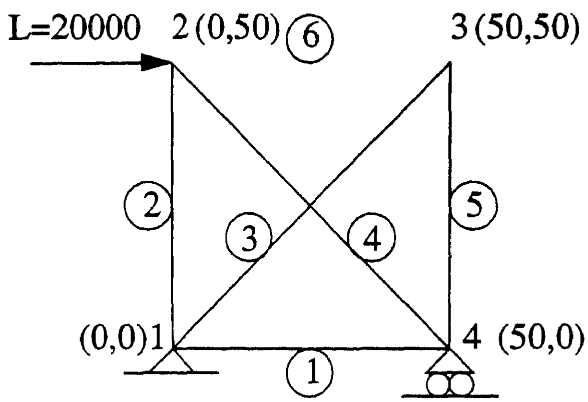
E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $ITER,1,1,1

AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	20000	tension
2	20000	tension
3	0	
4	-28284	compression
6	0	

APPENDIX G5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

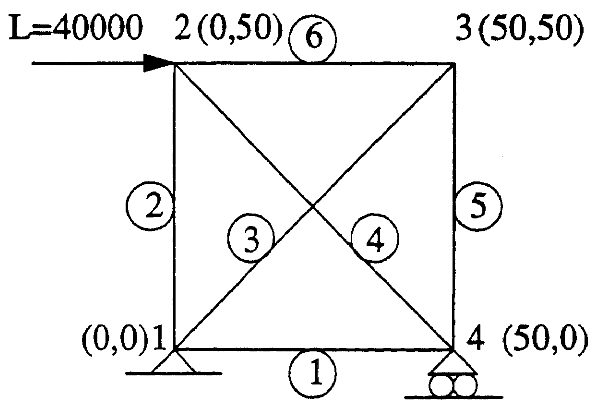


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,20000 $ITER,1,1,1 AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	20000	tension
2	20000	tension
3	0	
4	-28284	compression
5	0	

APPENDIX H ANSYS PROGRAM AND FORCE DISTRIBUTIONS

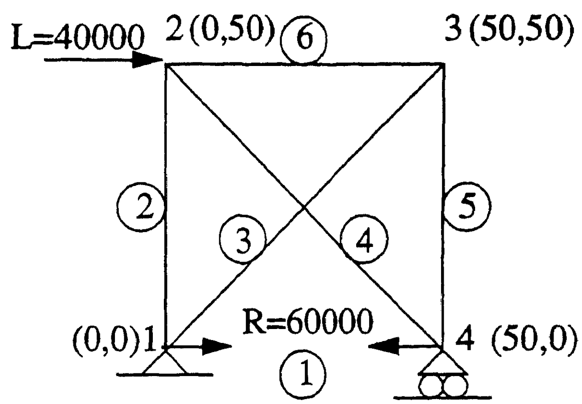


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,40000 ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	20000	tension
2	20000	tension
3	28284	tension
4	-28284	compression
5	-20000	compression
6	-20000	compression

APPENDIX H1 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

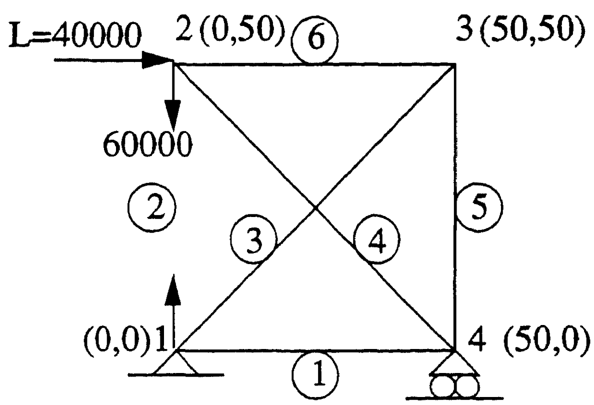


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0 E,1,2 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,40000 $F,1,FX,60000 F,4,FX,-60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
2	60000	tension
3	-28284	compression
4	-84853	compression
5	20000	tension
6	20000	tension

APPENDIX H2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

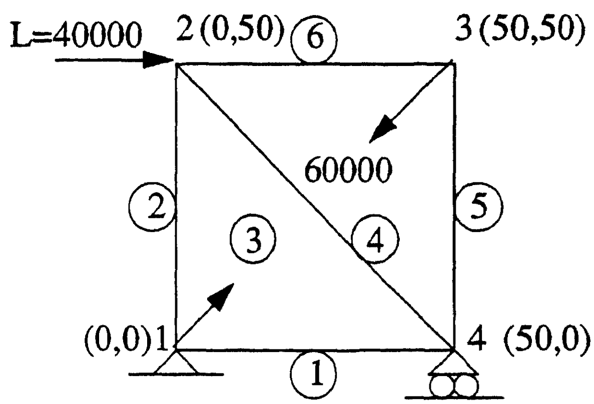


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,40000 $F,2,FY,-60000
F,1,FY,60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET
PRDISP $NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	60000	tension
3	-28284	compression
4	-84853	compression
5	20000	tension
6	20000	tension

APPENDIX H3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

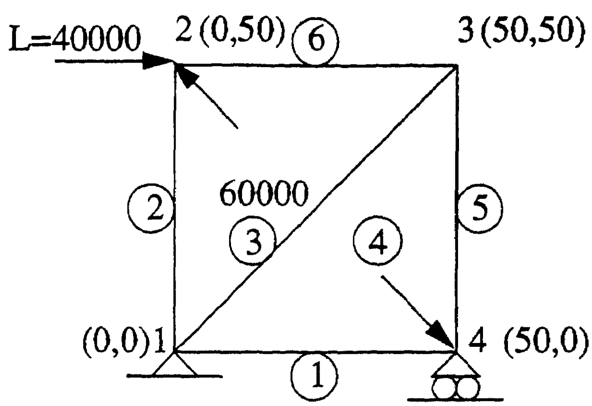


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $N,4,50,0
E,1,4 $E,1,2 $E,2,4 $E,3,4 $E,2,3 $EPLOT $D,1,ALL,0 $D,4,UY,0 $F,2,FX,40000
F,1,FX,42426.4 $F,1,FY,42426.4 $F,3,FX,-42426.4 $F,3,FY,-42426.4 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-2426.4	compression
2	-2426.4	compression
4	3431.4	tension
5	-42426	compression
6	-42426	compression

APPENDIX H4 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

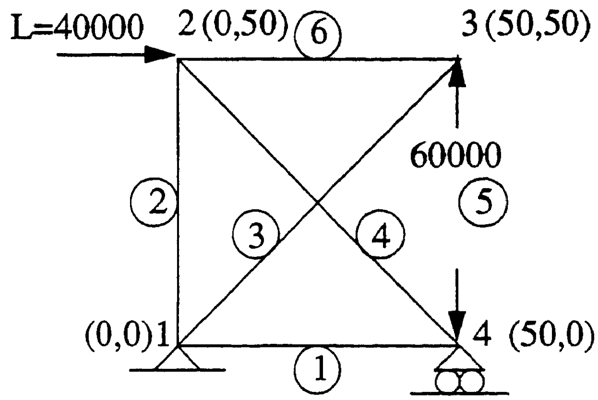


/PREP7 $KAN,0 $ET,1,1 $R,1,2 $MP,EX,1,30E6 $N,1 $N,2,0,50 $N,3,50,50 $$N,4,50,0
E,1,4 $E,1,2 $E,1,3 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,-2426.4
F,2,FY,42426.4 $F,4,FX,42426.4 $F,4,FY,-42426.4 $ITER,1,1,1 $AFWRITE $FINISH
/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE $FINISH

FORCE DISTRIBUTION:

	ELEMENT	FORCE
1	42426	tension
2	42426	tension
3	-3431.4	compression
5	2426.4	tension
6	2426.4	tension

APPENDIX H5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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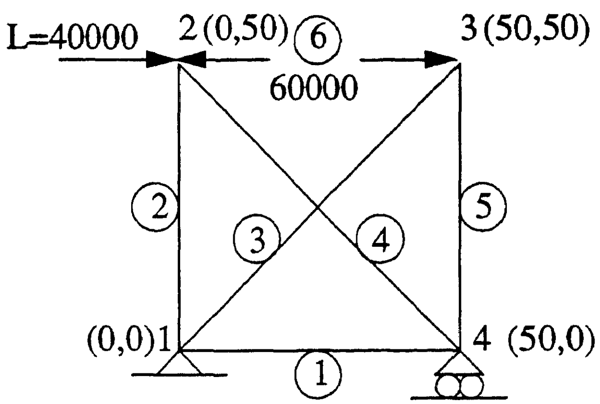
FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-20000	compression
2	-20000	compression
3	84853	tension
4	28284	tension
6	-60000	compression

APPENDIX H6 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,-20000 $F,3,FX,60000

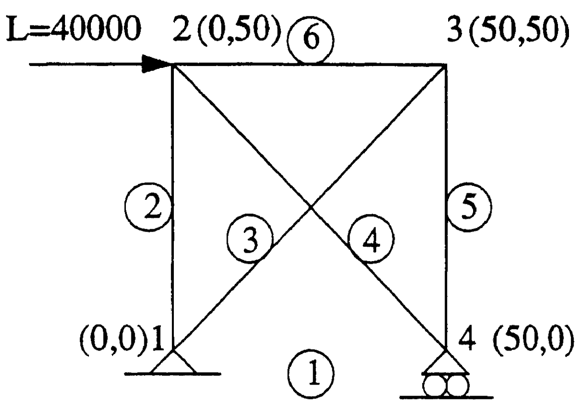
ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP

NFORCE $FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	-20000	compression
2	-20000	compression
3	84853	tension
4	28284	tension
5	-60000	compression

APPENDIX I ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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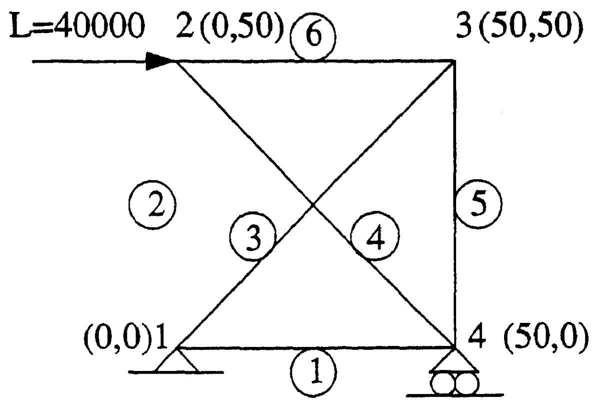
3	56569	tension
4	0	
5	-40000	compression
6	-40000	compression

APPENDIX I1 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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E,1,4 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,40000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTION:

	ELEMENT	FORCE
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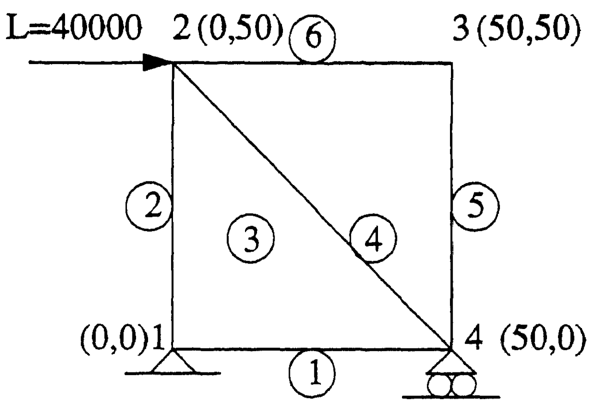
3	56569	tension
4	0	
5	-40000	compression
6	-40000	compression

APPENDIX I2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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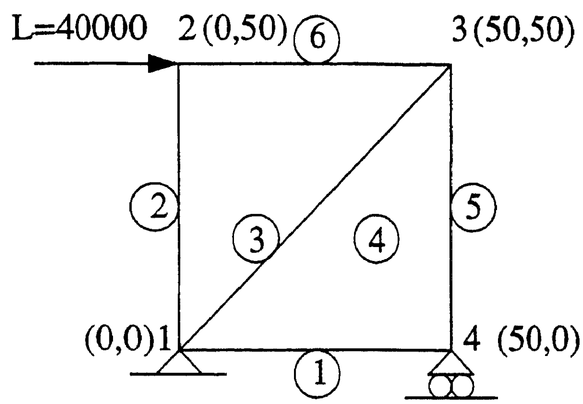
2	40000	tension
4	-56569	compression
5	0	
6	0	

APPENDIX I3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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E,1,4 $E,1,2 $E,1,3 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,40000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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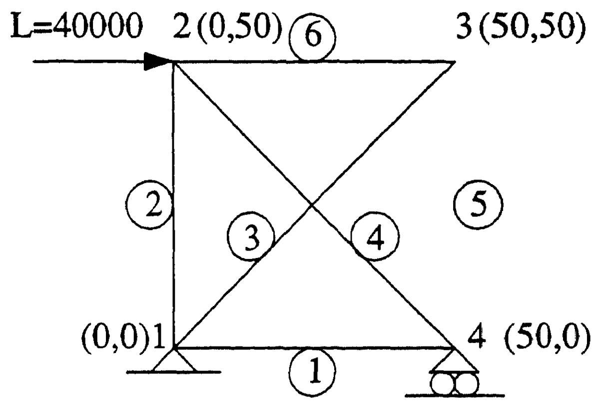
2	0	
3	56569	tension
5	-40000	compression
6	-40000	compression

APPENDIX I4 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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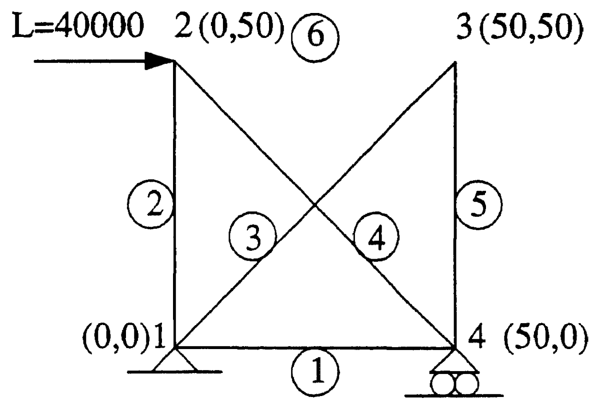
2	40000	tension
3	0	
4	-56569	compression
6	0	

APPENDIX I5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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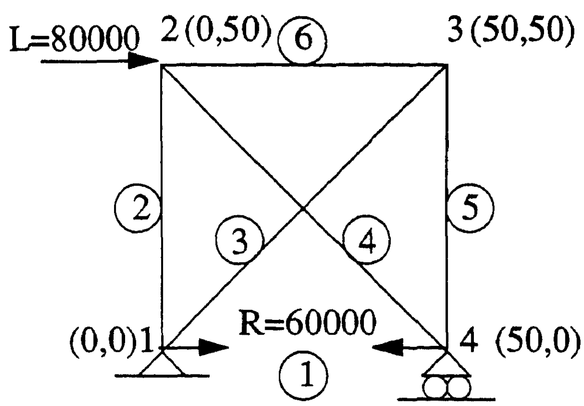
2	40000	tension
3	0	
4	-56569	compression
5	0	

APPENDIX J1 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

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F,4,FX,-60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET
PRDISP  $NFORCE $FINISH
```

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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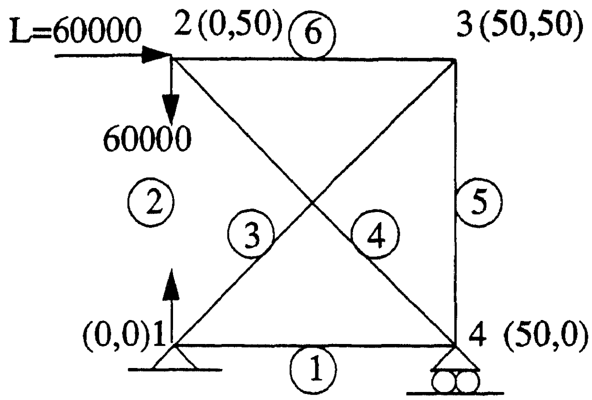
3	0	
4	-84853	compression
5	0	
6	0	

APPENDIX J2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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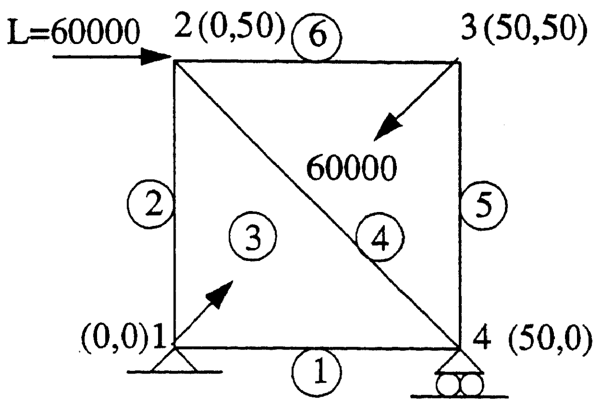
FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	60000	tension
3	0	
4	-84853	compression
5	0	
6	0	

APPENDIX J3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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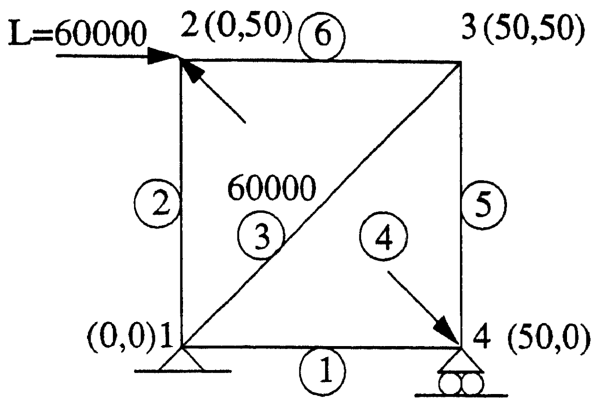
FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	17574	tension
2	17574	tension
4	-24853	compression
5	-42426	compression
6	-42426	compression

APPENDIX J4 ANSYS PROGRAM AND FORCE DISTRIBUTIONS

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F,2,FY,42426.4 $F,4,FX,42426.4 $F,4,FY,-42426.4 $ITER,1,1,1 $AFWRITE $FINISH
/INPUT,27 $FINISH $/POST1  $SET $PRDISP $NFORCE $FINISH
```
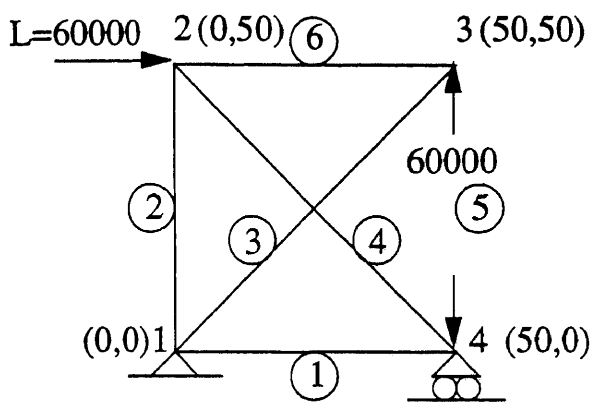

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	42426	tension
2	42426	tension
3	24853	tension
5	-17574	compression
6	-17574	compression

APPENDIX J5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,60000 $F,3,FY,60000
F,4,FY,-60000 $ITER,1,1,1 $AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET
PRDISP $NFORCE $FINISH

FORCE DISTRIBUTION:

	ELEMENT	FORCE
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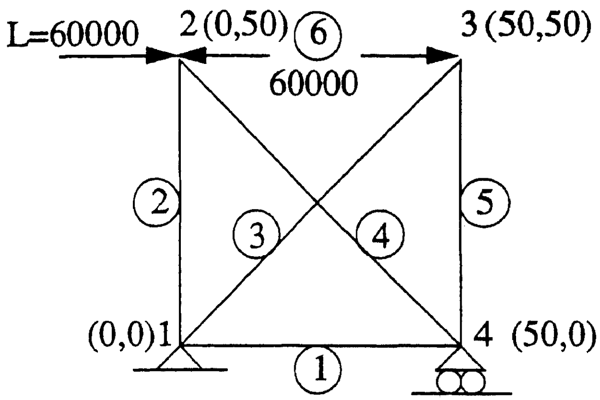
2	0	
3	84853	tension
4	0	
6	-60000	compression

APPENDIX J6 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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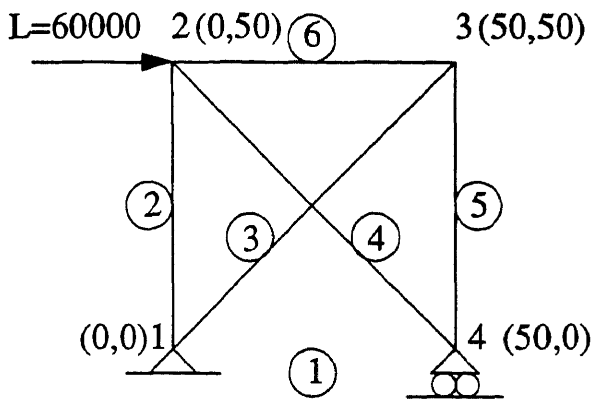
2	0	
3	84853	tension
4	0	
5	-60000	compression

APPENDIX K ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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E,1,2.$E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,60000 $ITER,1,1,1

AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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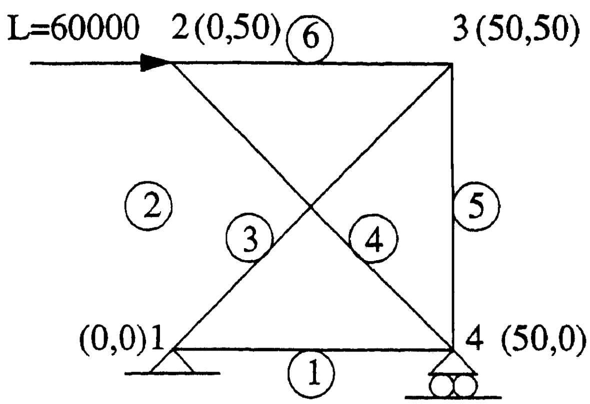
4	0	
5	-60000	compression
6	-60000	compression

APPENDIX K1 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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E,1,4 $E,1,3 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,60000 $ITER,1,1,1
AFWRITE $$FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE
FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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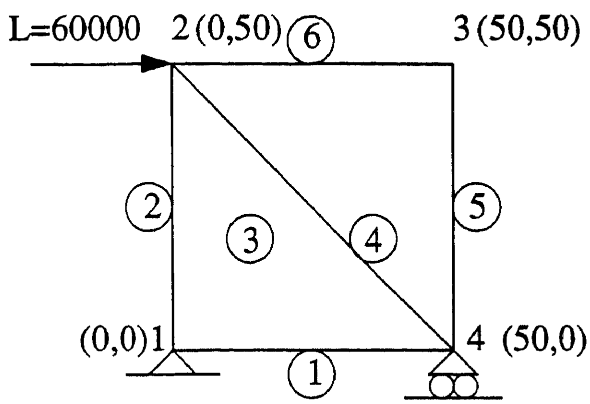
4	0	
5	-60000	compression
6	-60000	compression

APPENDIX K2 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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E,1,4 $E,1,2 $E,2,4 $E,3,4 $E,2,3 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,60000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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2	60000	tension
4	-84853	compression
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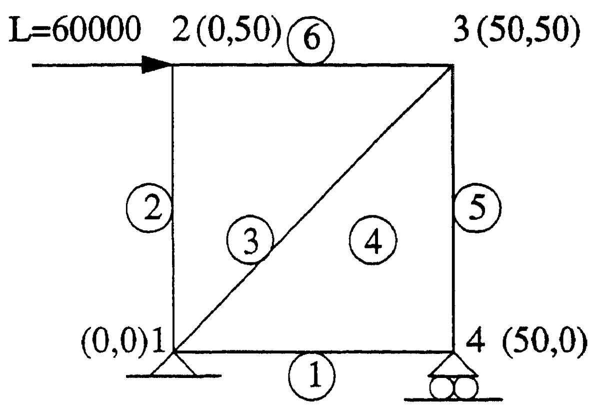
6	0	

APPENDIX K3 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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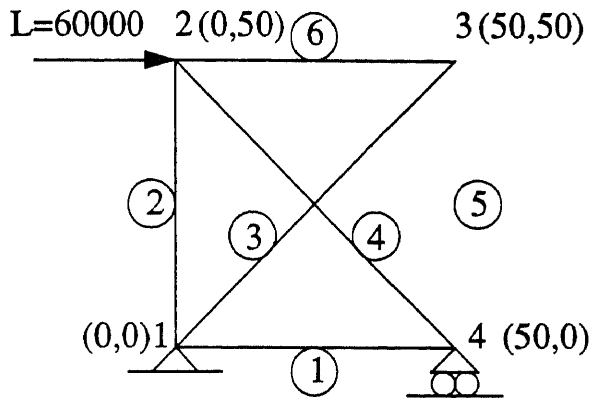
2	0	
3	84853	tension
5	-60000	compression
6	-60000	compression

APPENDIX K4 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
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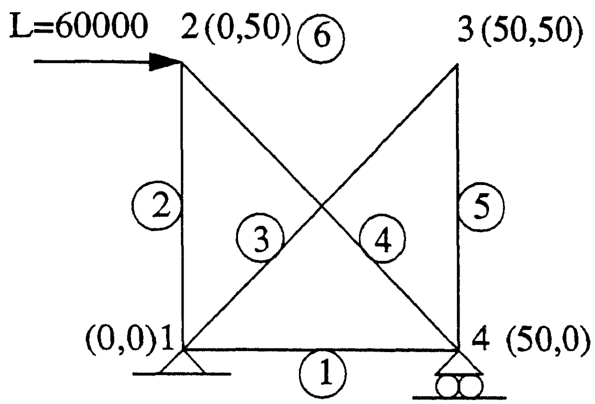
2	60000	tension
3	0	
4	-84853	compression
6	0	

APPENDIX K5 ANSYS PROGRAM AND FORCE DISTRIBUTIONS



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E,1,4 $E,1,2 $E,1,3 $E,2,4 $E,3,4 $D,1,ALL,0 $D,4,UY,0 $F,2,FX,60000 $ITER,1,1,1
AFWRITE $FINISH $/INPUT,27 $FINISH $/POST1 $SET $PRDISP $NFORCE FINISH

FORCE DISTRIBUTIONS:

	ELEMENT	FORCE
1	60000	tension
2	60000	tension
3	0	
4	-84853	compression
5	0	

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